Two Problems about Transversals in Multiplication Tables

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Problem 1: Isotopic Subsquares

Let $H \trianglelefteq Q$ be Moufang loops with $a, b \in Q$.

Problem: Are the $H \times H$ and $aH \times bH$ blocks isotopic as latin squares?

That is, are there bijections:

- $\alpha : H \to aH$
- $\beta : H \to bH$
- $\gamma : H \to (ab)H$

such that for all $h_1, h_2 \in H$,

$$\alpha(h_1)\beta(h_2) = \gamma(h_1 h_2)?$$

	1H				5H				9H				13H			
1H	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
	2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
	3	4	2	1	7	8	6	5	11	12	10	9	15	16	14	13
	4	3	1	2	8	7	5	6	12	11	9	10	16	15	13	14
5H	5	6	8	7	1	2	4	3	13	14	16	15	9	10	12	11
	6	5	7	8	2	1	3	4	14	13	15	16	10	9	11	12
	7	8	5	6	3	4	1	2	16	15	14	13	12	11	10	9
	8	7	6	5	4	3	2	1	15	16	13	14	11	12	9	10
9H	9	10	12	11	13	14	16	15	1	2	4	3	5	6	8	7
	10	9	11	12	14	13	15	16	2	1	3	4	6	5	7	8
	11	12	9	10	16	15	14	13	3	4	1	2	8	7	6	5
	12	11	10	9	15	16	13	14	4	3	2	1	7	8	5	6
12H	13	14	16	15	9	10	12	11	5	6	8	7	1	2	4	3
	14	13	15	16	10	9	11	12	6	5	7	8	2	1	3	4
	15	16	13	14	12	11	10	9	8	7	6	5	3	4	1	2
	16	15	14	13	11	12	9	10	7	8	5	6	4	3	2	1

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

$1H \times 1H$

	1	2	3	4
5	5	6	8	7
6	6	5	7	8
7	7	8	5	6
8	8	7	6	5

$5H \times 1H$

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

$1H \times 1H$

	1	2	3	4
5	5	6	8	7
6	6	5	7	8
7	7	8	5	6
8	8	7	6	5

$5H \times 1H$

To show that these squares are isotopic, we need bijections

- $\alpha : H \to 5H$
- $\beta : H \to H$
- $\gamma : H \to 5H$

such that for all $h_1, h_2 \in H$

$$\begin{aligned} \alpha(h_1)\beta(h_2) &= \gamma(h_1 h_2) \\ (5h_1)h_2 &= 5(h_1 h_2) \end{aligned}$$

... and this happens to be true in this loop.

1	2	3	4	5	6	7	8
2	4	5	6	7	1	8	3
3	5	1	7	2	8	4	6
4	6	7	1	8	2	3	5
5	7	2	8	4	3	6	1
6	1	8	2	3	4	5	7
7	8	4	3	6	5	1	2
8	3	6	5	1	7	2	4

9	10	11	12	13	14	15	16
10	12	13	14	15	9	16	11
11	13	9	15	10	16	12	14
12	14	15	9	16	10	11	13
13	15	10	16	12	11	14	9
14	9	16	10	11	12	13	15
15	16	12	11	14	13	9	10
16	11	14	13	9	15	10	12

Preliminary Results

Lemma

If $H \trianglelefteq Q$ are Moufang loops and $a \in Q$, then the following blocks are isotopic as latin squares:

1. $H \times H$
2. $aH \times aH$
3. $a^2 H \times a^{-1} H$

If $|Q| < 64$ and $a, b \in Q$, then $H \times H \approx aH \times bH$.

If Q is a CC-loop, then $H \times H \approx aH \times bH$ for all $a, b, \in Q$.
(Recent result of Kinyon - 15 minutes ago)

Question

Can we at least show that $H \times H \approx aH \times H$?

In groups, we can use the maps

$$\alpha(h_1)\beta(h_2) = \gamma(h_1 h_2)$$
$$(ah_1)h_2 = a(h_1 h_2)$$

Perhaps in Moufang loops we can have

$$\alpha(h_1)\beta(h_2) = \gamma(h_1 h_2)$$
$$(ah_1)h_2 = z$$

where z is a term on (a) and $(h_1 h_2)$.

But, Mace4 identifies the smallest non-associative Moufang loop as a counterexample (i.e. the value of the term $(ah_1)h_2$ is not a function of (a) and $(h_1 h_2)$).

Problem 2: Product of All Elements in a Loop

Definition

Let $(Q, \cdot)$ be a loop.

Let $P = \{q_1 \cdots q_n : q_i \in Q\}$ be the set of elements that can be written as a product containing every element of Q precisely once (with no use of the inverse operation).

Let $P^\omega = \{q_1 \cdots q_{kn} : q_i \in Q, k \in \mathbb{N}\}$ be the set of elements that can be written as a product containing every element of Q precisely k times for some $k \in \mathbb{N}$.

Theorem (Dénes, Hermann 1982)

If Q is a group, then P is a coset of Q'.

Their proof was highly non-trivial (i.e. used the Feit-Thompson theorem).

Conjecture

If Q is any loop, then P is a coset of Q'.

Theorem

If Q is an A-loop (= inner-mappings are automorphisms), then $P^\omega = Q'$ or $P^\omega = Q' \cup aQ'$ for some $a \in Q$.

How Prover9 / Mace4 might help...

- Find a counterexample to the conjecture that $P = aQ'$.
- Find a proof that $P^\omega \trianglelefteq Q$ (i.e. fixed by inner-mappings). This would imply that $P^\omega = Q'$ or $P^\omega = Q' \cup aQ'$.

The first difficulty in approaching either problem with Prover9/Mace4 is coding the property that $x \in P$.