Are Commutants in Moufang Loops Normal?

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$(((C * x) * y) * (y' * x')) * z = z * (((C * x) * y) * (y' * x'))$

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If you can't show that D is in the commutant, instead, show that D has many (some?) commutant-like properties; e.g., D^3 is nuclear (and piles and Piles and PILES of others).

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Theorem. (No assumptions on A, B, or C). (i) D commutes with cubes. (ii) If D commutes with E, and if D commutes with F, then D commutes with $E * F$ (so if L is generated by cubes, then D is in the commutant).