The AIM Problem in Loop Theory

Conjecture:

Let Q be an Abelian Inner Mapping (AIM) loop. Then $Q/N(Q)$ is an abelian group and $Q/Z(Q)$ is a group. In particular, Q is nilpotent of class at most 3.

M. Kinyon. `cl-informatik.uibk.ac.at/users/cek/aitp16/2016/slides/Kinyon_Obergurgl.pdf`, 2016.

M. Kinyon, R. Veroff and P. Vojtechovsky. Loops with Abelian Inner Mapping Groups: an Application of Automated Deduction. In M. P. Bonacina and M. Stickel, editors, *Automated Reasoning and Mathematics: Essays in Memory of William W. McCune, Lecture Notes in Artificial Intelligence* **7788**:151–164, Springer, 2013.

AIM Loops (Clauses)

```
% loop axioms
1 * x = x.              x * 1 = x.
x \ (x * y) = y.        x * (x \ y) = y.
(x * y) / y = x.        (x / y) * y = x.

% inner mappings
(y * x) \ (y * (x * u)) = L(u,x,y).
((u * x) * y) / (x * y) = R(u,x,y).
x \ (u * x) = T(u,x).

% abelian inner mapping group
T(T(u,x),y) = T(T(u,y),x).
L(L(u,x,y),z,w) = L(L(u,z,w),x,y).
R(R(u,x,y),z,w) = R(R(u,z,w),x,y).
T(L(u,x,y),z) = L(T(u,z),x,y).
T(R(u,x,y),z) = R(T(u,z),x,y).
L(R(u,x,y),z,w) = R(L(u,z,w),x,y).
```

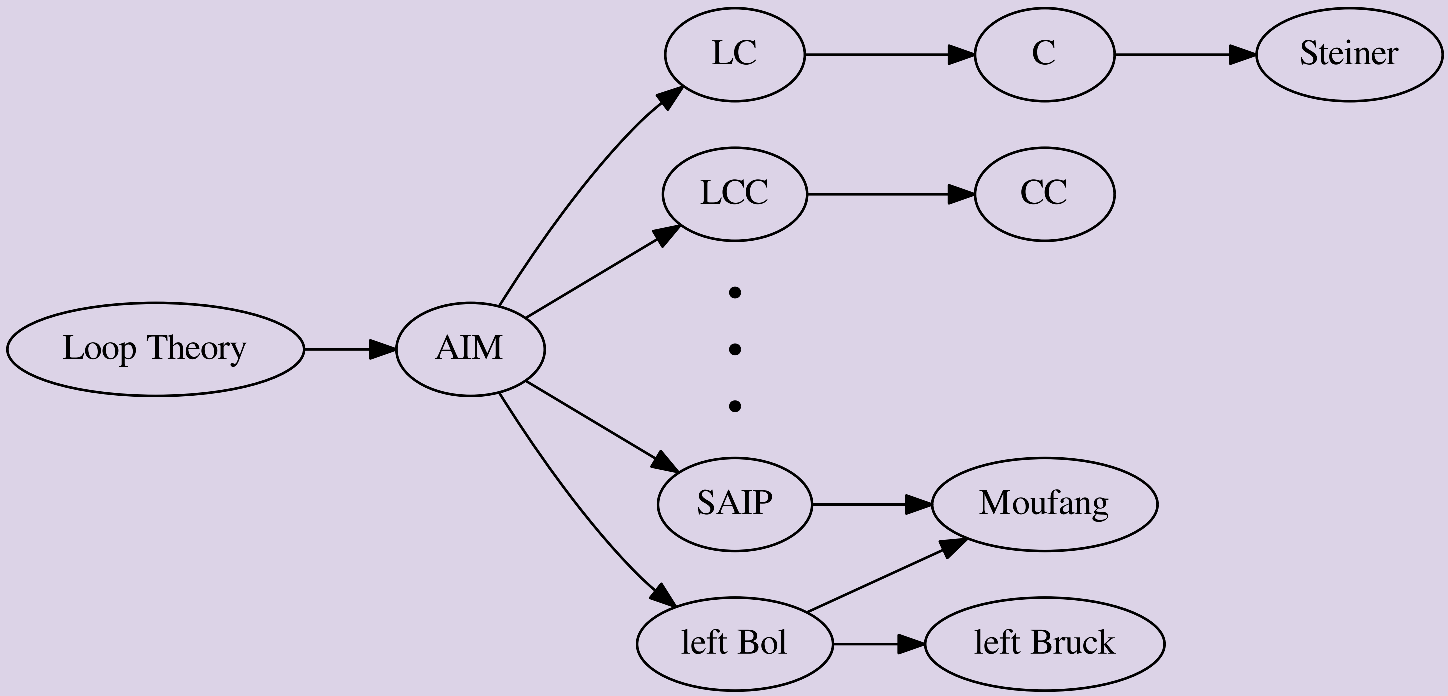
AIM Conjecture (Clauses)

```
% associator
(x * (y * z)) \ ((x * y) * z) = a(x,y,z).

% commutator
(x * y) \ (y * x) = K(y,x).

% goals
a(K(x,y),z,u) = 1    # label("aK1").
a(x,K(y,z),u) = 1    # label("aK2").
a(x,y,K(z,u)) = 1    # label("aK3").
K(a(x,y,z),u) = 1    # label("Ka").
a(a(x,y,z),u,w) = 1  # label("aa1").
a(x,a(y,z,u),w) = 1  # label("aa2").
a(x,y,a(z,u,w)) = 1  # label("aa3").
```

AIM Theory Hierarchy



The Challenges

How far up the loop hierarchy can we prove the conjecture?

Although general AIM is the ultimate goal, results in several of the extensions of the theory are new and of significant interest.

We also are interested in discovering other, previously unspecified, properties of AIM loops.

The Process

We boot-strapped the project (for initial sets of hints) by proving the conjecture in strong extensions of the theory for which the result was known (e.g., Moufang loops)

On success: move farther up the hierarchy

On failure:

- Prover9 parameters (but not a lot)
- additional and/or different extra assumptions
- intermediate lemmas (e.g., suggested by the mathematicians)
- looser or different characterization of related theorem
- iterative methods (e.g., varying lex order of terms)

As the library of proofs grows, it becomes increasingly important to manage hints (selection and prioritization) effectively.

Notable Results (Mathematics)

- Original goals in extensions of the theory (working up the hierarchy)
- Goal equivalences

  `aK1`, `aK2`, `aK3` and `Ka` are equivalent (in AIM)

  `aa1`, `aa2`, `aa3` are equivalent (in AIM)

  Proving `aK2` implies the others was *extremely* difficult

- Previously unknown properties of AIM loops, for example,

```
K(K(x,y),z) = K(x,K(y,z)).
a(x,y,z) * K(u,w) = K(u,w) * a(x,y,z).
```

See `www.cs.unm.edu/~veroff/AIM/`.

AIM Proof Lengths

2011:	2015:	2017:	2019:
24,356	73,625	242,134	242,134
18,862	69,489	141,589	193,847
17,075	54,742	112,135	141,589
16,400	45,131	89,716	124,938
15,785	40,708	87,534	112,135

Proof levels: Several over 500, one at 841

Other Measures of Progress

Before hint prioritization: 549 proofs in 117 output files, 167K distinct hint clauses, 47K appearing in more than one output file

As of November 2018: 641 proofs in 149 output files, 2.3 million distinct hints, 90K appearing in more than two output files

As of January 2019: 660 proofs in 158 output files, 2.6 million distinct hint clauses, 114K appearing in more than two output files

Other Projects

Lattices, groups, loops, classical and nonclassical logics ...

Publications in respected math journals: Algebra Universalis, Journal of Algebra, Transactions of the AMS, Notre Dame Journal of Formal Logic, Studia Logica ...