CS 422/522: Image and Pattern Analysis
Homework 3 (Spring '05)

1. Characterize the following as linear, or non-linear, and as even, odd, or neither, and prove your characterization. In these functions, x is real, and z is complex.

   (a) $f(x) = 7x + 1$

   (b) $f(x) = x^2 - x$

   (c) $f(x) = e^{-x^2}$

   (d) $f(z) = 3z - 2$

   (e) $f(z) = z - z^*$

   (f) $f(z) = \mathrm{Im}(z)$

   (g) $f(z) = z^{\frac{1}{2}}$

2. The $\int$ operator takes a function, f, as its argument and returns the anti-derivative of the function: $f \xrightarrow{\int} \int f(t)dt$. Prove that the $\int$ operator is:

   (a) Linear.

   (b) Shift-invariant.

3. Prove that $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$.

4. The impulse response function of a linear, shift-invariant system is:

   $$h(t) = \frac{\sin(\pi t)}{\pi t}$$

   and its input is:

   $$x(t) = \cos(4\pi t) + \cos(\pi t/2).$$

   What is its output?

5. The impulse response function of a linear, shift-invariant system is:

   $$h(t) = e^{-\frac{\pi t^2}{2}}$$

   and its input is:

   $$x(t) = e^{j2\pi s_0 t}.$$

   What is its output?

6. The sine Gabor function is the product of a sine and a Gaussian, $f(t) = e^{-\pi t^2}\sin(2\pi s_0 t)$. Give an expression for $F(s)$, the Fourier transform of $f(t)$.

7. Prove that the sum of two Gaussian random variables with variances σ_1^2 and σ_2^2 is a Gaussian random variable with variance $\sigma_1^2 + \sigma_2^2$.

8. The function, $f(t)$, is defined as:

$$f(t) = \begin{cases} 1 & \text{if } |at - b| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

   Give an expression for $F(s)$, the Fourier transform of $f(t)$.

9. The transfer function of a linear shift invariant system is $H(s) = 1/s$. The impulse response function, $h(t)$, is $\mathcal{F}^{-1}\{H(s)\}$. Give an expression for $g(t)$ where:

$$g(t) = \int_{-\infty}^{\infty} e^{j2\pi s_0 \tau} h(t-\tau)\, d\tau.$$

10. Compute the Fourier transform of $f(t) = -2\pi t\, e^{-\pi t^2}\cos(2\pi s_0 t)$. Hint: What is $\frac{d\left(e^{-\pi t^2}\right)}{dt}$?

11. Prove the following statement: If $\mathcal{F}\{f\}(s) = F(s)$ then $\mathcal{F}\{F\}(s) = f(-s)$. Hint: If $\mathcal{F}\{f\}(s) = F(s)$ then $\mathcal{F}^{-1}\{F\}(t) = f(t)$.

12. Prove that $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$