Two Channel Subband Coding

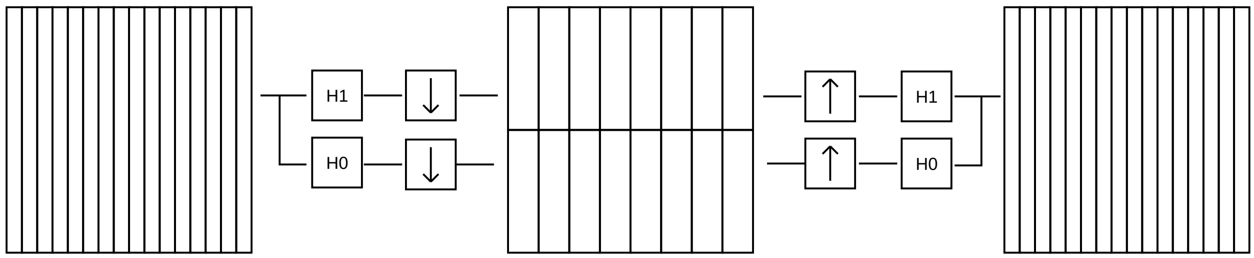


Figure 1: Two channel subband coding.

In two channel subband coding

- A signal is convolved with a highpass filter $\vec{h}_1$ and a lowpass filter $\vec{h}_0$.
- The two halfband signals are then downsampled.
- These operations trade one bit of resolution in time for one bit of resolution in frequency.

The process can be inverted by

- Upsampling each halfband signal
- Convolving each halfband signal with a time reversed filter
- Adding the results.

Discrete Orthogonal Wavelet Design

- Given a highpass filter, $\vec{h}_1$, of length, $N = 8$, with four non-zero taps, a, b, c, and d.
- If the inner product of

$$\vec{h}_1 = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & a & b \end{bmatrix}^{\mathrm{T}}$$

and samples of a constant function, $f(t) = 1$, is zero:

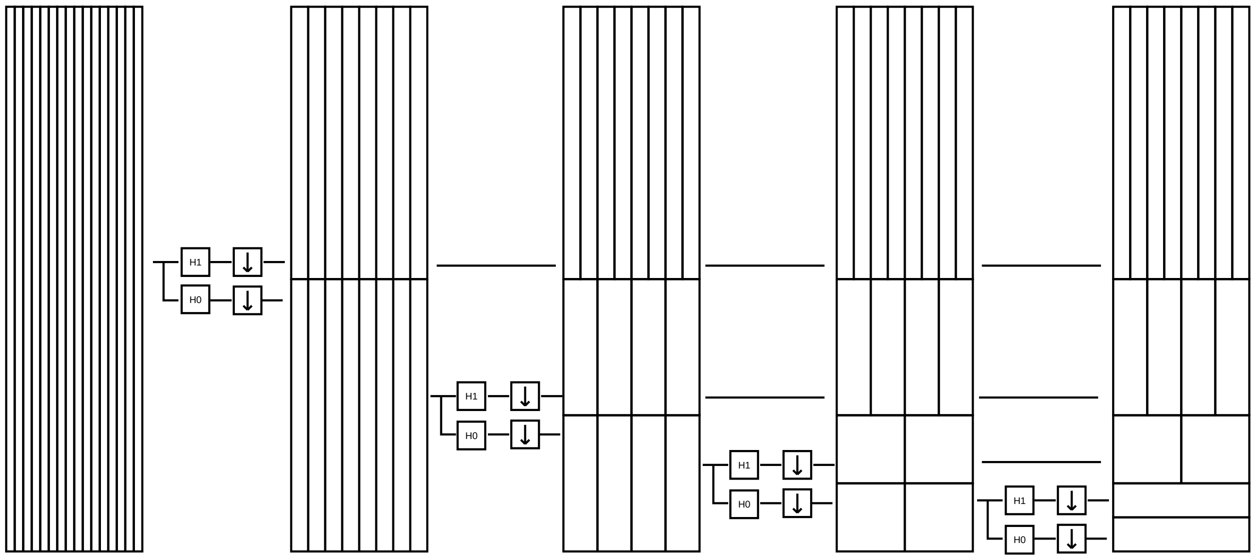
$$a \cdot 1 + b \cdot 1 + c \cdot 1 + d \cdot 1 = 0$$

then $\vec{h}_1$ has *one vanishing moment*.

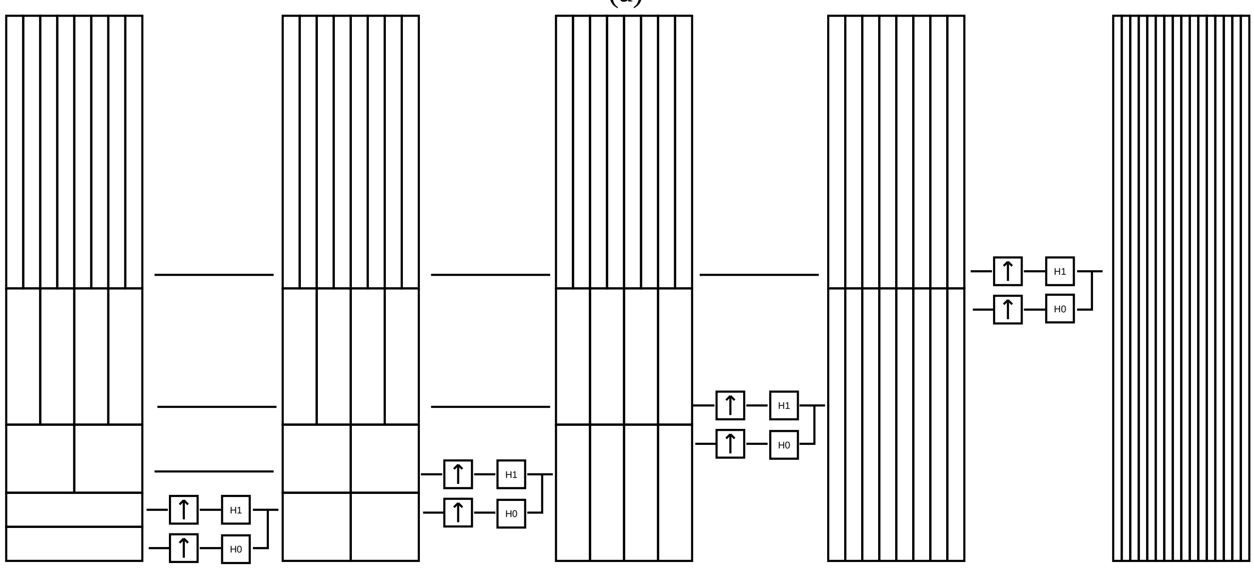
- If (additionally) the inner product of $\vec{h}_1$ and samples of a linear ramp function, $f(t) = t$, is zero:

$$a \cdot (-2) + b \cdot (-1) + c \cdot 0 + d \cdot 1 = 0$$

then $\vec{h}_1$ has *two vanishing moments*.



(a)



(b)

Figure 2: (a) Recursive application of two channel subband coding to the lower halfband signal results in *diadic* sampling of time and frequency. This process is called the *fast wavelet transform*. (b) The process can be inverted to recover the original signal.

Discrete Orthogonal Wavelet Design (contd.)

We also require that $\vec{h}_1$ be orthogonal to all of its even shifts.

$$\begin{pmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \end{pmatrix} = \begin{pmatrix} c & d & 0 & 0 & 0 & 0 & a & b \\ a & b & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$

This means that the taps must satisfy two additional constraints:

$$a \cdot a + b \cdot b + c \cdot c + d \cdot d = 1$$

and

$$a \cdot c + b \cdot d = 0.$$

Alternating Flip with Odd Shift

The lowpass filter, $\vec{h}_0$, is created from the high-pass filter, $\vec{h}_1$, as follows:

$$\vec{h}_0(n) = (-1)^n \vec{h}_1(K-n)$$

This combines the following three operations:

- Reflect.
- Shift by an odd amount.
- Alternate signs.

$\vec{h}_0$ and $\vec{h}_1$ are termed *conjugate mirror filters*.

Example

Given a highpass filter $\vec{h}_1$ of length $N = 8$:

$$\vec{h}_1 = \begin{bmatrix} c & d & 0 & 0 & 0 & 0 & a & b \end{bmatrix}^{\mathrm{T}}.$$

1. Reflect $\vec{h}_1$ about the origin to get:

$$\vec{h}_1(-n) = \begin{bmatrix} c & b & a & 0 & 0 & 0 & 0 & d \end{bmatrix}^{\mathrm{T}}.$$

2. Shift it by $K = 1$ to get:

$$\vec{h}_1(1-n) = \begin{bmatrix} b & a & 0 & 0 & 0 & 0 & d & c \end{bmatrix}^{\mathrm{T}}.$$

3. Alternate the signs to get the taps of the lowpass filter $\vec{h}_0$:

$$\begin{aligned} \vec{h}_0 &= (-1)^n \vec{h}_1(1-n) \\ &= \begin{bmatrix} b & -a & 0 & 0 & 0 & 0 & d & -c \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

Two Channel Subband Coding (contd.)

The two channel subband coding matrix looks like this:

$$\begin{pmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ \ell_1 \\ \ell_3 \\ \ell_5 \\ \ell_7 \end{pmatrix} = \begin{pmatrix} c & d & 0 & 0 & 0 & 0 & a & b \\ a & b & c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c & d \\ b & -a & 0 & 0 & 0 & 0 & d & -c \\ d & -c & b & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & d & -c & b & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & d & -c & b & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix}$$

We observe that

- $\vec{h}_0$ is orthogonal to all even shifts of $\vec{h}_0$
- $\vec{h}_0$ is orthogonal to all even shifts of $\vec{h}_1$
- $\vec{h}_1$ is orthogonal to all even shifts of $\vec{h}_1$.

The Daubechies 4 Wavelet

The values, $a = \frac{1-\sqrt{3}}{4\sqrt{2}}$, $b = -\frac{3-\sqrt{3}}{4\sqrt{2}}$, $c = \frac{3+\sqrt{3}}{4\sqrt{2}}$, $d = -\frac{1+\sqrt{3}}{4\sqrt{2}}$ satisfy the following constraints:

- $a \cdot a + b \cdot b + c \cdot c + d \cdot d = 1$
- $a \cdot c + b \cdot d = 0$
- $a \cdot 1 + b \cdot 1 + c \cdot 1 + d \cdot 1 = 0$
- $a \cdot (-2) + b \cdot (-1) + c \cdot 0 + d \cdot 1 = 0$

It follows that the Daubechies 4 highpass filter has two vanishing moments.

Orthonormal Wavelet Series

Recall that the daughter wavelets and the mother wavelet in a dyadic wavelet series transform are related as follows:

$$\Psi_{j,k}(x) = \frac{1}{\sqrt{2^j}}\Psi\left(\frac{x-k2^j}{2^j}\right).$$

We seek a mother wavelet Ψ where the daughter wavelets $\Psi_{j,k}$ for $-\infty \leq j \leq \infty$ and $-\infty \leq k \leq \infty$ form an orthonormal basis for the space of square integrable functions ("The Holy Grail"):

- **Analysis**

$$< f, \Psi_{j,k} >= \int_{-\infty}^{\infty} f(x)\overline{\Psi_{j,k}(x)}dx$$

- **Synthesis**

$$f(x) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} < f, \Psi_{j,k} > \Psi_{j,k}(x).$$

Orthonormal Wavelet Series

A discrete signal can be represented by a vector, $\vec{h}$. However, it can also be represented by a continuous signal, $h(.)$, equal to a weighted sum of shifted impulses:

$$h(t) = \sum_{i=-\infty}^{\infty} \vec{h}(i)\, \delta(t-i).$$

We can model convolution and downsampling of discrete signals using continuous representations. By the sifting property, the convolution of a continuous signal, $g(.)$, and a continuous representation of a discrete filter, $h(.)$, is:

$$\{g * h\}(t) = \sum_{i=-\infty}^{\infty} \vec{h}(i)\, g(t-i).$$

Orthonormal Wavelet Series (contd.)

The effect of downsampling a discrete signal is modeled by dilating its continuous representation, $g(.)$, by a factor of one-half:

$$g(t) \rightarrow g(2t).$$

The combined effects of convolving a continuous signal, $g(.)$, with a continuous representation of a discrete filter, $h(.)$, and downsampling is then:

$$\{g * h\}(2t) = \sum_{i=-\infty}^{\infty} \vec{h}(i)\, g(2t - i).$$

Orthonormal Wavelet Series (contd.)

The *scaling function* (or “father”) is the *fixed point* of the lowpass filtering and downsampling operations:

$$\Phi(t) = \sum_{i=-\infty}^{\infty} \vec{h}_0(i)\,\Phi(2t-i).$$

The *wavelet* (or “mother”) is derived from the scaling function by a single highpass filtering and downsampling operation:

$$\Psi(t) = \sum_{i=-\infty}^{\infty} \vec{h}_1(i)\,\Phi(2t-i).$$

Orthonormal Wavelet Series (contd.)

The daughter wavelets

$$\Psi_{j,k}(x) = \frac{1}{\sqrt{2^j}}\Psi\left(\frac{x - k2^j}{2^j}\right)$$

where $-\infty \leq j \leq \infty$ and $-\infty \leq k \leq \infty$ form an orthonormal basis for the space of square integrable functions!

The Daubechies 4 Wavelet (contd.)

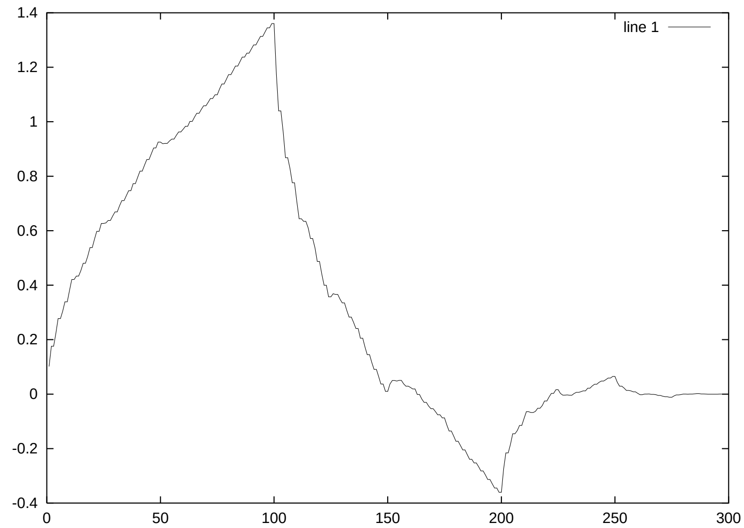


Figure 3: Daubechies 4 scaling function, Φ.

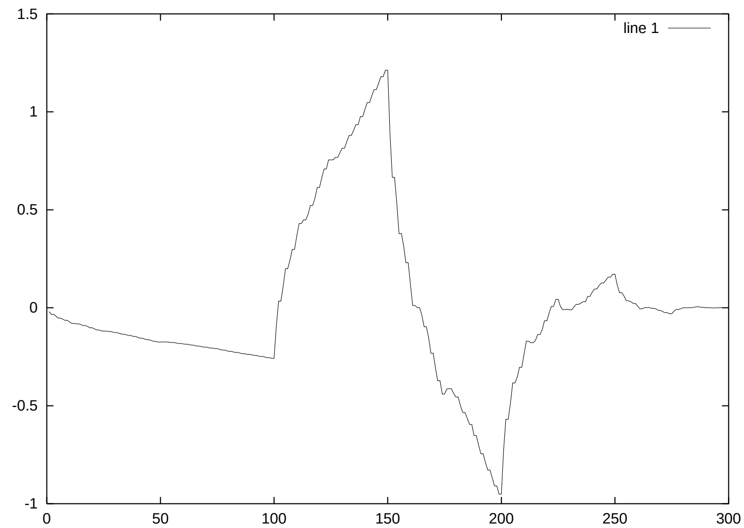


Figure 4: Daubechies 4 wavelet, Ψ.

Conjugate Mirror Filters

We have seen that the alternating flip with odd shift can be used to find an orthogonal h_0. But how do we know that h_0 is lowpass? We want the amplitudes of the transfer functions to be equal except for a shift by $N/2$:

$$|H_0(m)| = |H_1(m + N/2)|$$

This will guarantee that h_0 is lowpass if h_1 high-pass.

Conjugate Mirror Filters (contd.)

Shifting and reflection (conjugation) have *no* effect on the *amplitude* of H_1 (they affect only its *phase*):

$$\begin{aligned} |\mathcal{F}\{h_1(n)\}| &= |H_1(m)| \\ &= \left|\overline{e^{-j2\pi m\frac{K}{N}} H_1(m)}\right| \\ &= |\mathcal{F}\{h_1(K-n)\}|. \end{aligned}$$

We conclude that $h_1(n)$ and $h_1(K-n)$ have the same power spectrum.

Conjugate Mirror Filters (contd.)

What effect does the $N/2$ shift have on the impulse response function?

$$
\begin{aligned}
\mathcal{F}^{-1}\{H_1(m+N/2)\} &= e^{-j2\pi n\frac{N/2}{N}} h_1(n) \\
&= e^{-j\pi n} h_1(n) \\
&= (-1)^n h_1(n)
\end{aligned}
$$

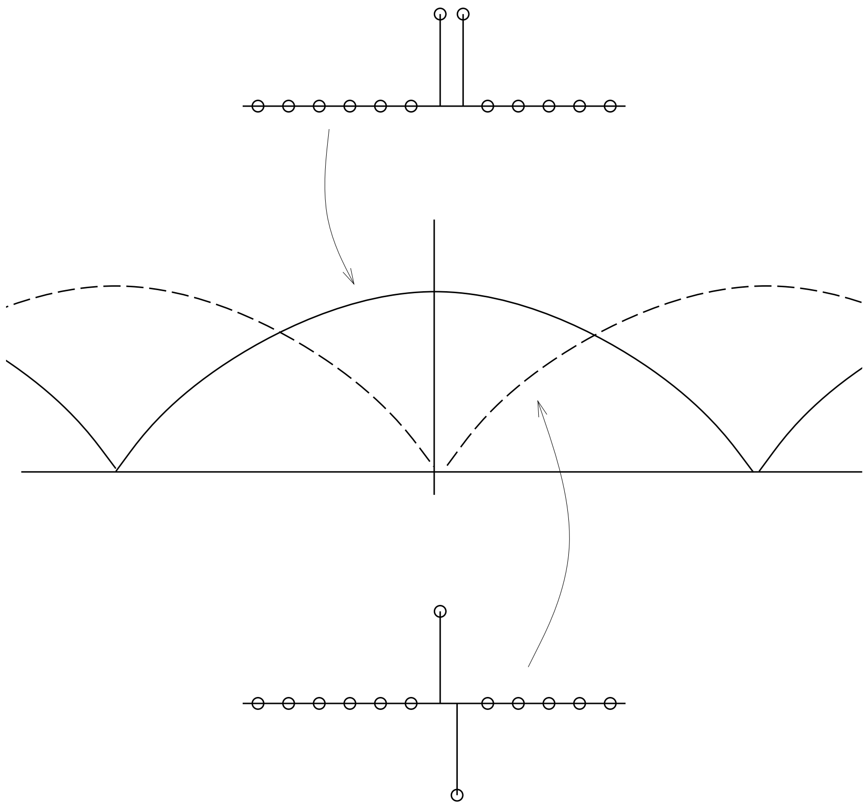


Figure 5: The Haar lowpass filter, h_0, and highpass filter, h_1, and their Fourier transform amplitudes.

Conjugate Mirror Filters (contd.)

We now see where each of the three steps came from:

- Conjugation in frequency domain is reflection in space domain.
- Shift by $N/2$ in frequency domain is achieved by changing signs of odd coefficients in space domain.
- Multiplication by $e^{-j2\pi mK/N}$ in frequency domain is shift by K in space domain.

Comment: The fact that one can simultaneously achieve orthogonality and complementarity (in the lowpass/highpass sense) by such a simple manipulation is pretty amazing!

Conjugate Mirror Filters (contd.)

Let's look at what two channel subband coding looks like in the frequency domain:

- **Analysis**

$$
\begin{aligned}
F_0(m) &= \overline{H_0(m)}F(m) \\
F_1(m) &= \overline{H_1(m)}F(m)
\end{aligned}
$$

- **Synthesis**

$$F(m) = F_0(m)H_0(m) + F_1(m)H_1(m)$$

Conjugate Mirror Filters (contd.)

Substituting the analysis expressions for F_0 and F_1 into the synthesis expression yields:

$$F(m) = F(m)\overline{H_0(m)}H_0(m) + F(m)\overline{H_1(m)}H_1(m)$$

which means that

$$F(m) = F(m)\left[|H_0(m)|^2 + |H_1(m)|^2\right],$$

so that

$$|H_0(m)|^2 + |H_1(m)|^2 = 1$$

which can be solved for the transfer function of the lowpass filter:

$$|H_0(m)|^2 = 1 - |H_1(m)|^2.$$

Thus, an appropriate highpass filter, *i.e.*, a filter with the desired number of vanishing moments, is all that is required to design a discrete orthogonal wavelet transform.