Space-Frequency Atoms

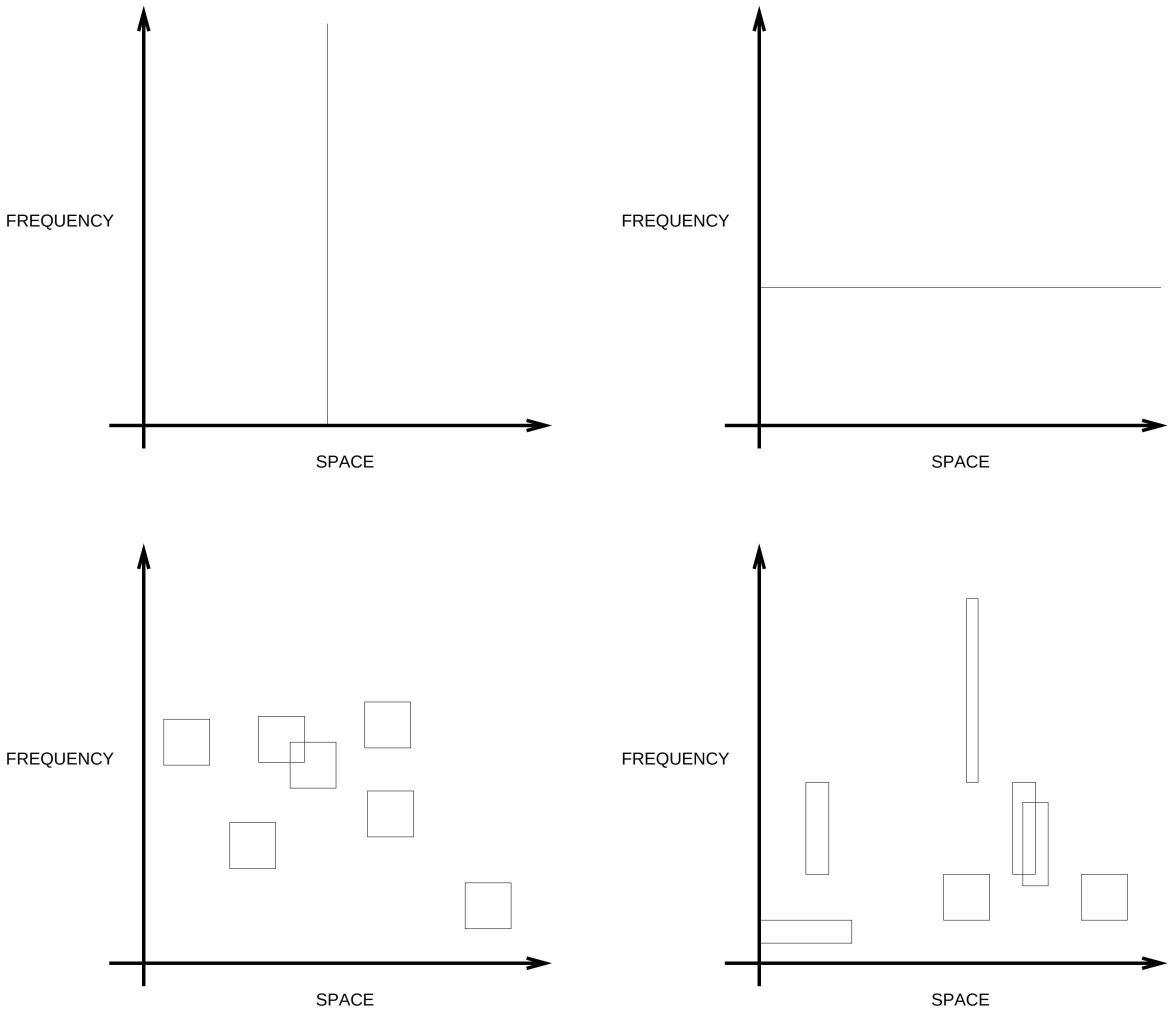

Figure 1: Space-frequency atoms.

Windowed Fourier Transform

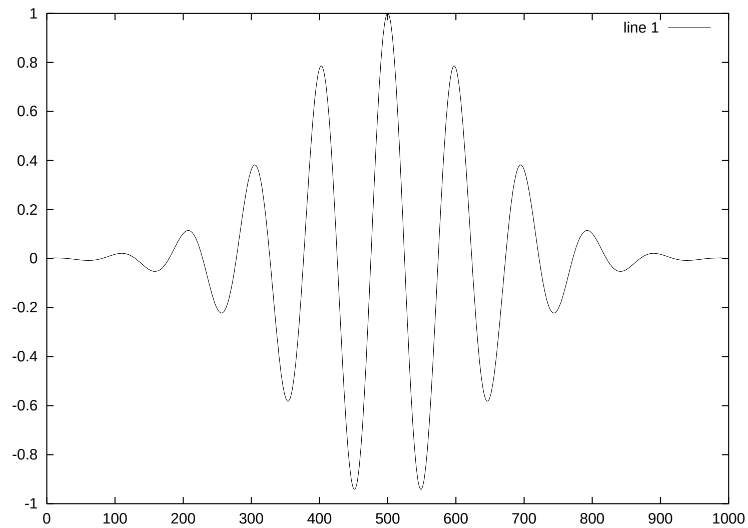

Figure 2: A Gabor function.

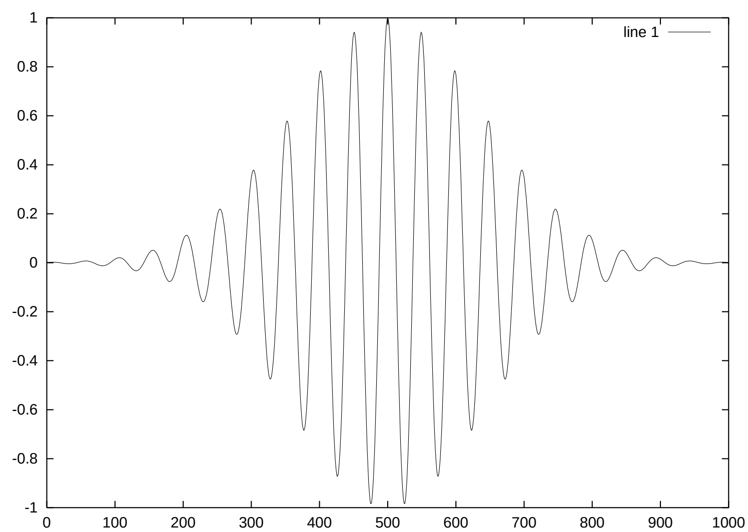

Figure 3: A second Gabor function.

Windowed Fourier Transform (contd.)

- Analysis

$$F(u,b) = \langle w(x-b)e^{j2\pi ux},\ f\,\rangle = \int_{-\infty}^{\infty} f(x)w(x-b)e^{-j2\pi ux}\,dx$$

- Synthesis

$$f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,b)w(x-b)e^{j2\pi ux}\,du\,db$$

What is a Wavelet?

All basis functions (daughter wavelets) are generated by *translation* and *dilation* of a *mother* wavelet:

$$\Psi_{a,b}(x) = \frac{1}{\sqrt{a}}\Psi\left(\frac{x-b}{a}\right)$$

when $a < 1$ it shrinks the wavelet. The $\sqrt{a}$ factor keeps the norm constant:

$$\left\|f\left(\frac{x-b}{a}\right)\right\| = \sqrt{\int_{-\infty}^{\infty}\left|f\left(\frac{x-b}{a}\right)\right|^2 dx}$$
$$= \sqrt{a}\|f(x)\|.$$

What is a Wavelet? (contd.)

The mother wavelet, Ψ, must satisfy the *admissibility criterion*:

$$C_{\Psi} = \int_{-\infty}^{\infty} \frac{|\widehat{\Psi}(s)|^2}{|s|} ds < \infty$$

where $\widehat{\Psi}$ is the Fourier transform of Ψ. This means that:

- $|\widehat{\Psi}(s)|^2$ decays faster than $1/|s|$
- $\widehat{\Psi}(0) = 0$.

What is a Wavelet? (contd.)

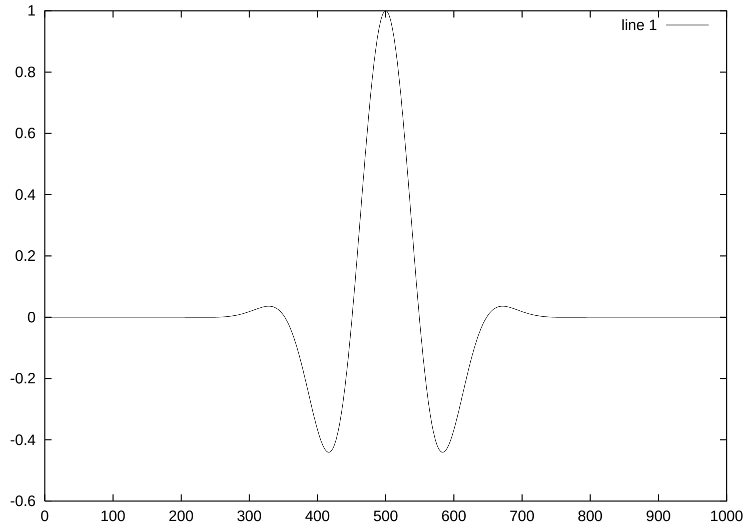

Figure 4: A Morlet wavelet.

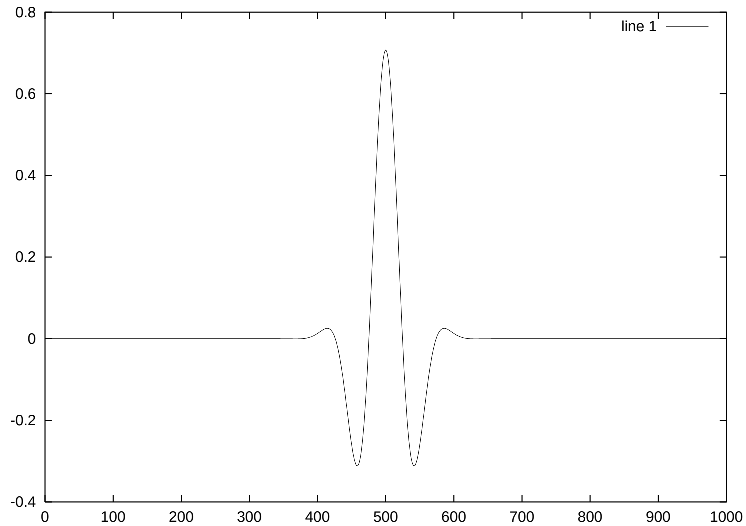

Figure 5: A second Morlet wavelet.

Vanishing Moments

- The n-th moment of Ψ is defined to be:
$$M_n\{\Psi\} = \int_{-\infty}^{\infty} t^n \, \Psi(t) dt.$$
- If $M_0\{\Psi\} = 0$ then Ψ has one *vanishing moment.*
- Because
$$M_0\{\Psi\} = \int_{-\infty}^{\infty} \Psi(x) dx = \widehat{\Psi}(0) = 0$$
all wavelets have at least one vanishing moment.
- If $M_0\{\Psi\} = M_1\{\Psi\} = 0$, then Ψ has two vanishing moments, etc.

Vanishing Moments (contd.)

- If Ψ has one vanishing moment, then

$$\langle \Psi_{a,b},\ a_0 \rangle = 0.$$

- If Ψ has two vanishing moments, then

$$\langle \Psi_{a,b},\ a_1 x + a_0 \rangle = 0.$$

- If Ψ has n vanishing moments, then

$$\langle \Psi_{a,b},\ a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \rangle = 0,$$

  i.e., the daughter wavelets are orthogonal to any polynomial of degree less than n.

- Vanishing moments are the reason why smooth signals have sparse representations in wavelet bases.

Continuous Wavelet Transform

- **Analysis**

$$\begin{aligned} F(a,b) &= \langle \Psi_{a,b},\ f \rangle \\ &= \int_{-\infty}^{\infty} f(x)\overline{\Psi_{a,b}(x)}\, dx \end{aligned}$$

- **Synthesis**

$$f(x) = \frac{1}{C_\Psi}\int_0^\infty \frac{1}{a^2}\int_{-\infty}^\infty F(a,b)\Psi_{a,b}(x)\, db\, da$$

where

$$C_\Psi = \int_{-\infty}^\infty \frac{|\widehat{\Psi}(s)|^2}{|s|} ds$$

Continuous Wavelet Transform (Example)

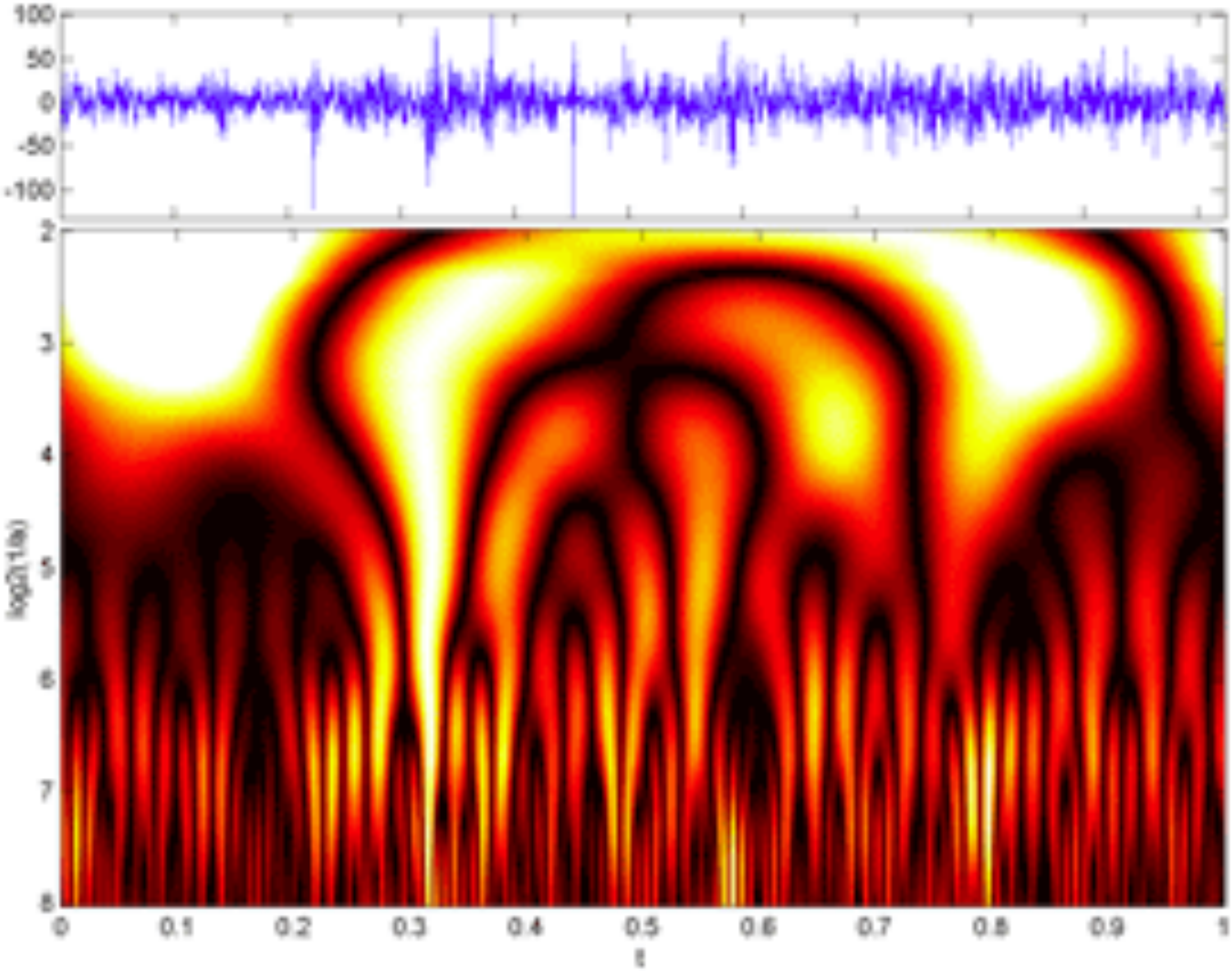

Figure 6: Continuous wavelet transform of time-series using derivative of Gaussian wavelet (from Vialar, T., *Complex and Chaotic Nonlinear Dynamics*, Springer, 2009).

Two Dimensional Continuous Wavelet Transform

- Analysis

$$F(a,b_x,b_y) = \langle \Psi_{a,b_x,b_y},\ f \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\overline{\Psi_{a,b_x,b_y}(x,y)}\,dx\,dy$$

- Synthesis

$$f(x,y) = \frac{1}{C_\Psi}\int_0^{\infty}\frac{1}{a^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(a,b_x,b_y)\Psi_{a,b_x,b_y}(x,y)\,db_x\,db_y\,da$$

where

$$\Psi_{a,b_x,b_y}(x,y) = \frac{1}{|a|}\Psi\left(\frac{x-b_x}{a},\frac{y-b_y}{a}\right)$$

and

$$C_\Psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\widehat{\Psi}(u,v)|^2}{\sqrt{|u|^2+|v|^2}}dudv$$

Wavelet Transform as Convolution

Recall that the relationship between daughter wavelet $\Psi_{a,b}$ and mother wavelet Ψ involves both translation and dilation:

$$\Psi_{a,b}(x) = \frac{1}{\sqrt{a}}\Psi\left(\frac{x-b}{a}\right).$$

Let's define a function Ψ_a to represent a daughter which is dilated by a factor a but is not translated:

$$\Psi_a(x-b) = \Psi_{a,b}(x) = \frac{1}{\sqrt{a}}\Psi\left(\frac{x-b}{a}\right)$$

and a function $\overline{\Psi}_a(x)$ to represent a reflected and conjugated instance of Ψ_a:

$$\overline{\Psi}_a(x) = \overline{\Psi_a(-x)}.$$

Wavelet Transform as Convolution (contd.)

Using Ψ_a and $\overline{\Psi}_a$ the forward and inverse continuous wavelet transforms can be expressed as follows:

- Analysis

$$\begin{aligned} F(a,b) &= \langle \Psi_{a,b},\ f \rangle \\ &= \int_{-\infty}^{\infty} f(x)\overline{\Psi_a(x-b)}\,dx \\ &= \int_{-\infty}^{\infty} f(x)\overline{\Psi}_a(b-x)\,dx \\ &= \{f * \overline{\Psi}_a\}(b) \end{aligned}$$

- Synthesis

$$\begin{aligned} f(x) &= \frac{1}{C_\Psi}\int_0^{\infty}\frac{1}{a^2}\int_{-\infty}^{\infty}\langle \Psi_{a,b},\ f \rangle\,\Psi_{a,b}(x)\,db\,da \\ &= \frac{1}{C_\Psi}\int_0^{\infty}\frac{1}{a^2}\int_{-\infty}^{\infty}\{f * \overline{\Psi}_a\}(b)\Psi_a(x-b)\,db\,da \\ &= \frac{1}{C_\Psi}\int_0^{\infty}\frac{1}{a^2}\{f * \overline{\Psi}_a * \Psi_a\}(x)\,da \end{aligned}$$

<u>Wavelet Series Transform</u>

Is it possible to replace the integrals over a and b in the synthesis formula with sums? Can we represent any f in a Hilbert space, $\mathcal{H}$, using a discrete set, S, of wavelet coefficients? If for all $f \in \mathcal{H}$ there exist $A > 0$ and $B < \infty$ such that

$$A||f||^2 \leq \sum_{(a,b)\in S} |\langle \Psi_{a,b},\ f \rangle|^2 \leq B||f||^2$$

then $\Psi_{a,b}$ for $(a,b) \in S$ form a frame for $\mathcal{H}$. Furthermore, there exists a set of functions $\widetilde{\Psi}_{a,b}$ for $(a,b) \in S$ which form a *dual frame* for $\mathcal{H}$:

$$\frac{1}{B}||f||^2 \leq \sum_{(a,b)\in S} |\langle \widetilde{\Psi}_{a,b},\ f \rangle|^2 \leq \frac{1}{A}||f||^2.$$

Wavelet Series Transform (contd.)

The wavelets, $\Psi_{a,b}$, are used for analysis:

$$\langle \Psi_{a,b},\ f \rangle = \int_{-\infty}^{\infty} f(x)\overline{\Psi_{a,b}(x)}\, dx$$

and the wavelets, $\widetilde{\Psi}_{a,b}$, are used for synthesis:

$$f(x) = \sum_{(a,b)\in S} \langle \Psi_{a,b},\ f \rangle \widetilde{\Psi}_{a,b}(x).$$

Self-inverting Wavelet Series

If $A = B$, then

$$\sum_{(a,b)\in S} |\langle \Psi_{a,b},\ f\ \rangle|^2 = A||f||^2$$

and the $\Psi_{a,b}$ for $(a,b) \in S$ form a *tight-frame* for $\mathcal{H}$, in which case

$$f(x) = \frac{1}{A} \sum_{(a,b)\in S} \langle \Psi_{a,b},\ f\ \rangle \Psi_{a,b}(x).$$

Such frames are said to be *self-inverting* because $\Psi_{a,b}(x) = \frac{1}{A}\widetilde{\Psi}_{a,b}(x)$.

Redundancy

Recall that for a tight-frame

$$A = \frac{\sum_{(a,b)\in S} |\langle \Psi_{a,b},\ f\ \rangle|^2}{||f||^2}.$$

Assuming that $||\Psi|| = 1$, then A provides a measure of the redundancy of the expansion, i.e., the degree of *overcompleteness*. If $A = 1$ there is no redundancy, and the expansion is *orthonormal*. How can one find wavelet series transforms with no redundancy?

Dyadic Sampling

A sampling pattern is *dyadic* if the daughter wavelets are generated by dilating the mother wavelet by 2^j and translating it by $k2^j$:

$$\Psi_{j,k}(x) = \frac{1}{\sqrt{2^j}}\Psi\left(\frac{x - k2^j}{2^j}\right)$$

Dyadic sampling is optimal because the space variable is sampled at the Nyquist rate for any given frequency.

Dyadic Sampling (contd.)

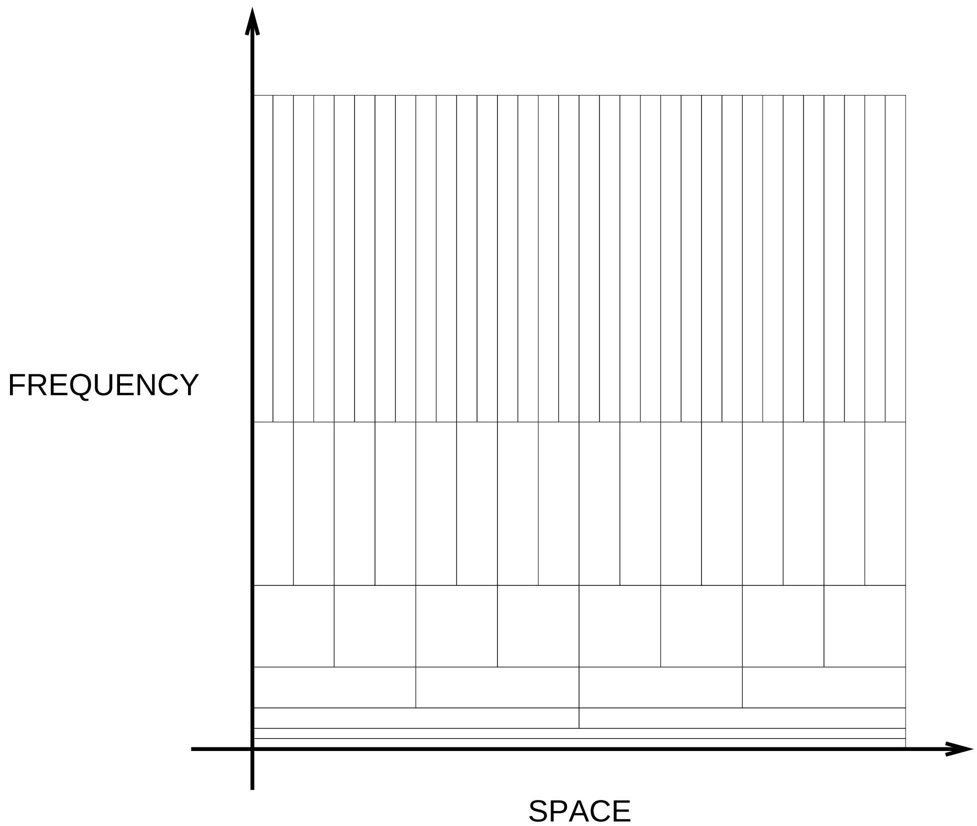

Figure 7: Dyadic sampling pattern.