Autocorrelation Function

$$
\begin{aligned}
R_f(\tau) &= \{f(t) * f(-t)\}(\tau) \\
&= \int_{-\infty}^{\infty} f(t) f(t+\tau)\, dt \\
&= \langle f(t) f(t+\tau) \rangle
\end{aligned}
$$

Power Spectrum

$$
\begin{aligned}
P_f(s) &= \mathcal{F}\left\{f(t) * f(-t)\right\}(s) \\
&= F(s)F(-s) \\
&= F(s)F^*(s) \\
&= |F(s)|^2
\end{aligned}
$$

Cross-correlation Function

$$
\begin{aligned}
R_{fg}(\tau) &= \{f(t) * g(-t)\}(\tau) \\
&= \int_{-\infty}^{\infty} f(t)g(t+\tau)\, dt
\end{aligned}
$$

Cross Power Spectrum

$$
\begin{aligned}
P_{fg}(s) &= \mathcal{F}\{f(t) * g(-t)\}(s) \\
&= F(s)G(-s) \\
&= F(s)G^*(s)
\end{aligned}
$$

Mean Squared Error

The input x of a linear shift invariant system with impulse response g is the sum of signal s and noise n:

$$\begin{aligned}\hat{s}(t) &= \int_{-\infty}^{\infty} g(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} g(\tau)\left[s(t-\tau)+n(t-\tau)\right]d\tau.\end{aligned}$$

The error signal resulting from the use of g is the difference between s and $\hat{s}$:

$$e\{g\}(t) = s(t) - \hat{s}(t)$$

The mean squared error is

$$\left\langle e^2\{g\}\right\rangle = \left\langle s^2 - 2s\hat{s} + \hat{s}^2\right\rangle$$

where $\langle f\rangle = \int_{-\infty}^{\infty} f(t)dt$ which is just

$$\left\langle e^2\{g\}\right\rangle = \left\langle s^2\right\rangle - 2\left\langle s\hat{s}\right\rangle + \left\langle \hat{s}^2\right\rangle$$

because $\langle . \rangle$ is linear.

Mean Squared Error (contd.)

The expected values in the expression for *MSE* can be defined in terms of correlation functions:

$$\langle s^2 \rangle \;=\; \int_{-\infty}^{\infty} s(t)s(t+0)\,dt = R_s(0)$$

$$\begin{aligned}
\langle s\,\hat{s} \rangle \;&=\; \int_{-\infty}^{\infty} s(t) \int_{-\infty}^{\infty} x(u)g(t-u)\,du\,dt \\
&=\; \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} s(t)x(t-u)\,dt\,du \\
&=\; \int_{-\infty}^{\infty} g(-u)R_{xs}(-u)\,du \\
&=\; \int_{-\infty}^{\infty} g(v)R_{xs}(v)\,dv
\end{aligned}$$

$$\begin{aligned}
\langle \hat{s}^2 \rangle \;&=\; \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u)x(t-u)\,du \int_{-\infty}^{\infty} g(v)x(t-v)dv \right) dt \\
&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u)g(v) \int_{-\infty}^{\infty} x(t-u)x(t-v)\,dt\,du\,dv \\
&=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u)g(v)R_x(u-v)\,du\,dv.
\end{aligned}$$

Mean Squared Error (contd.)

Substituting these expressions into the expression for $\left\langle e^2\{g\}\right\rangle$ yields

$$\begin{aligned}\left\langle e^2\{g\}\right\rangle &= R_s(0) - 2\int_{-\infty}^{\infty} g(\tau) R_{xs}(\tau)\, d\tau \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau) g(t) R_x(t-\tau)\, dt\, d\tau.\end{aligned}$$

Minimization

To show that x is a local minimum of f it suffices to show that $f(x) \leq f(x+\Delta x)$ for all Δx.

Functional Minimization

To show that f is a local minimum of $\left\langle e^2\right\rangle$ it suffices to show that $\left\langle e^2\{f\}\right\rangle \leq \left\langle e^2\{f+\delta f\}\right\rangle$ for all δf.

Minimization of Mean Squared Error

We will show that there exists an h such that

$$\underbrace{\left\langle e^2\{h\}\right\rangle}_{MSE_o} \leq \underbrace{\left\langle e^2\{h+\delta h\}\right\rangle}_{MSE}$$

for all δh. Letting $g = h + \delta h$ yields

$$\begin{aligned} MSE &= \underbrace{R_s(0) - 2\int_{-\infty}^{\infty} h(\tau)R_{xs}(\tau)\,d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)h(t)R_x(t-\tau)\,dt\,d\tau}_{MSE_o} \\ &+ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t)\delta h(\tau)R_x(t-\tau)\,dt\,d\tau \\ &- 2\int_{-\infty}^{\infty} \delta h(\tau)R_{xs}(\tau)\,d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta h(\tau)\delta h(t)R_x(t-\tau)\,dt\,d\tau. \end{aligned}$$

where the first three terms are MSE_o.

Minimization of Mean Squared Error (contd.)

Combining the fourth and fifth terms yields

$$\begin{aligned} MSE &= MSE_o + 2\int_{-\infty}^{\infty} \delta h(\tau) \left[\int_{-\infty}^{\infty} h(t) R_x(t-\tau)\, dt - R_{xs}(\tau) \right] d\tau \\ &+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta h(\tau) \delta h(t) R_x(t-\tau)\, dt\, d\tau. \end{aligned}$$

It is possible to show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta h(\tau) \delta h(t) R_x(t-\tau)\, dt\, d\tau =$$

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \delta h(\tau) x(t-\tau)\, d\tau \right]^2 dt \geq 0$$

for all δh. Consequently

$$MSE \geq MSE_o + 2\int_{-\infty}^{\infty} \delta h(\tau) \left[\int_{-\infty}^{\infty} h(t) R_x(t-\tau)\, dt - R_{xs}(\tau) \right] d\tau.$$

Wiener-Hopf Equation

We now observe that when the *Wiener-Hopf equation*

$$R_{xs}(\tau) = \int_{-\infty}^{\infty} h(t) R_x(t-\tau)$$

is satisfied that

$$\begin{aligned} MSE &\geq MSE_o + 2\int_{-\infty}^{\infty} \delta h(\tau) \underbrace{\left[\int_{-\infty}^{\infty} h(t) R_x(t-\tau)\, dt - R_{xs}(\tau)\right]}_{0} d\tau \\ &\geq MSE_o. \end{aligned}$$

It follows that h satisfying the Wiener-Hopf equation is the optimal linear filter.

Uncorrelated Signal and Noise

Assuming that s and n are uncorrelated:

$$\begin{aligned}
R_{xs}(t) &= \int_{-\infty}^{\infty} x(\tau)s(\tau+t)d\tau \\
&= \int_{-\infty}^{\infty} [s(\tau)+n(\tau)]\, s(\tau+t)d\tau \\
&= \int_{-\infty}^{\infty} s(\tau)s(\tau+t)d\tau + \underbrace{\int_{-\infty}^{\infty} n(\tau)s(\tau+t)d\tau}_{0} \\
&= \int_{-\infty}^{\infty} s(\tau)s(\tau+t)d\tau \\
&= R_s(t)
\end{aligned}$$

$$\begin{aligned}
R_x(t) &= \int_{-\infty}^{\infty} x(\tau)x(\tau+t)d\tau \\
&= \int_{-\infty}^{\infty} [s(\tau)+n(\tau)]\,[s(\tau+t)+n(\tau+t)]\, d\tau \\
&= \int_{-\infty}^{\infty} s(\tau)s(\tau+t)d\tau + \int_{-\infty}^{\infty} n(\tau)n(\tau+t)d\tau \\
&= R_s(t)+R_n(t)
\end{aligned}$$

Wiener Filter Transfer Function

Substituting the above expressions into the Wiener-Hopf equation results in

$$R_s(t) = \int_{-\infty}^{\infty} h(t-\tau)\,[R_s(\tau)+R_n(\tau)]\,d\tau.$$

Taking the Fourier transform of both sides yields

$$P_s(s) = H(s)\cdot[P_s(s)+P_n(s)]$$

where $P_s(s)$ and $P_n(s)$ are the power spectra of signal and noise. This equation can be solved for the transfer function of the optimal linear filter:

$$H(s) = \frac{P_s(s)}{P_s(s)+P_n(s)}.$$

Wiener Filter Impulse Response Function

$$h(t) = \mathcal{F}^{-1}\left\{\frac{P_s(s)}{P_s(s)+P_n(s)}\right\}(t)$$

Wiener Filter Mean Square Error

$$MSE_o = \int_{-\infty}^{\infty} P_n(s)h(s)\,ds$$