Fourier Transform Pairs

The Fourier transform transforms a function of time, $f(t)$, into a function of frequency, $F(s)$:

$$\mathcal{F}\{f(t)\}(s) = F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}dt.$$

The inverse Fourier transform transforms a function of frequency, $F(s)$, into a function of time, $f(t)$:

$$\mathcal{F}^{-1}\{F(s)\}(t) = f(t) = \int_{-\infty}^{\infty} F(s)e^{j2\pi st}ds.$$

The inverse Fourier transform of the Fourier transform is the identity transform:

$$f(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\tau)e^{-j2\pi s\tau}d\tau\right] e^{j2\pi st}ds.$$

Fourier Transform Pairs (contd).

Because the Fourier transform and the inverse Fourier transform differ only in the sign of the exponential's argument, the following reciprocal relation holds between $f(t)$ and $F(s)$:

$$f(t) \xrightarrow{\mathcal{F}} F(s)$$

is equivalent to

$$F(t) \xrightarrow{\mathcal{F}} f(-s).$$

This relationship is often written more economically as follows:

$$f(t) \xleftrightarrow{\mathcal{F}} F(s)$$

where $f(t)$ and $F(s)$ are said to be a *Fourier transform pair.*

Fourier Transform of Gaussian

Let $f(t)$ be a Gaussian:

$$f(t) = e^{-\pi t^2}.$$

By the definition of Fourier transform we see that:

$$\begin{aligned} F(s) &= \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi st} dt \\ &= \int_{-\infty}^{\infty} e^{-\pi(t^2 + j2st)} dt. \end{aligned}$$

Now we can multiply the right hand side by $e^{-\pi s^2} e^{\pi s^2} = 1$:

$$\begin{aligned} F(s) &= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t^2 + j2st) + \pi s^2} dt \\ &= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t^2 + j2st - s^2)} dt \\ &= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t + js)(t + js)} dt \\ &= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t + js)^2} dt \end{aligned}$$

Fourier Transform of Gaussian (contd.)

$$F(s) \;=\; e^{-\pi s^2}\int_{-\infty}^{\infty} e^{-\pi(t+js)^2} dt$$

After substituting u for $t + js$ and du for dt we see that:

$$F(s) = e^{-\pi s^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2} du}_{1}.$$

It follows that the Gaussian is its own Fourier transform:

$$e^{-\pi t^2} \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-\pi s^2}.$$

Fourier Transform of Dirac Delta Function

To compute the Fourier transform of an impulse we apply the definition of Fourier transform:

$$\mathcal{F}\left\{\delta(t-t_0)\right\}(s) = F(s) = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi st} dt$$

which, by the sifting property of the impulse, is just:

$$e^{-j2\pi s t_0}.$$

It follows that:

$$\delta(t-t_0) \xrightarrow{\mathcal{F}} e^{-j2\pi s t_0}.$$

Fourier Transform of Harmonic Signal

What is the inverse Fourier transform of an impulse located at s_0? Applying the definition of inverse Fourier transform yields:

$$\mathcal{F}^{-1}\{\delta(s-s_0)\}(t) = f(t) = \int_{-\infty}^{\infty} \delta(s-s_0) e^{j2\pi st} ds$$

which, by the sifting property of the impulse, is just:

$$e^{j2\pi s_0 t}.$$

It follows that:

$$e^{j2\pi s_0 t} \xrightarrow{\mathcal{F}} \delta(s-s_0).$$

Fourier Transform of Sine and Cosine

We can compute the Fourier transforms of the sine and cosine by exploiting the sifting property of the impulse:

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0).$$

- **Question** What is the inverse Fourier transform of a pair of impulses spaced symmetrically about the origin?

$$\mathcal{F}^{-1}\{\delta(s+s_0)+\delta(s-s_0)\}$$

- **Answer** By definition of inverse Fourier transform:

$$f(t) = \int_{-\infty}^{\infty} [\delta(s+s_0)+\delta(s-s_0)]\, e^{j2\pi st} ds.$$

Fourier Transform of Sine and Cosine (contd.)

Expanding the above yields the following expression for $f(t)$:

$$\int_{-\infty}^{\infty} \delta(s+s_0) e^{j2\pi st} ds + \int_{-\infty}^{\infty} \delta(s-s_0) e^{j2\pi st} ds$$

Which by the sifting property is just:

$$\begin{aligned} f(t) &= e^{j2\pi s_0 t} + e^{-j2\pi s_0 t} \\ &= 2\cos(2\pi s_0 t). \end{aligned}$$

Fourier Transform of Sine and Cosine (contd.)

It follows that

$$\cos(2\pi s_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left[\delta(s+s_0)+\delta(s-s_0)\right].$$

A similar argument can be used to show:

$$\sin(2\pi s_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{j}{2}\left[\delta(s+s_0)-\delta(s-s_0)\right].$$

Fourier Transform of the Pulse

To compute the Fourier transform of a pulse we apply the definition of Fourier transform:

$$\begin{aligned}
F(s) &= \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi st} dt \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi st} dt \\
&= \frac{1}{-j2\pi s} e^{-j2\pi st} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= \frac{1}{-j2\pi s} \left(e^{-j\pi s} - e^{j\pi s}\right) \\
&= \frac{1}{\pi s} \frac{\left(e^{j\pi s} - e^{-j\pi s}\right)}{2j}
\end{aligned}$$

Using the fact that $\sin(x) = \frac{\left(e^{jx} - e^{-jx}\right)}{2j}$ we see that:

$$F(s) = \frac{\sin(\pi s)}{\pi s}.$$

Fourier Transform of the Shah Function

Recall the Fourier series for the Shah function:

$$\frac{1}{2\pi}\text{III}\left(\frac{t}{2\pi}\right) = \frac{1}{2\pi}\sum_{\omega=-\infty}^{\infty} e^{j\omega t}.$$

By the sifting property,

$$\text{III}\left(\frac{t}{2\pi}\right) = \sum_{\omega=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(s-\omega)e^{jst}ds.$$

Changing the order of the summation and the integral yields

$$\text{III}\left(\frac{t}{2\pi}\right) = \int_{-\infty}^{\infty}\sum_{\omega=-\infty}^{\infty}\delta(s-\omega)e^{jst}ds.$$

Factoring out e^{jst} from the summation

$$\begin{aligned}\text{III}\left(\frac{t}{2\pi}\right) &= \int_{-\infty}^{\infty} e^{jst}\sum_{\omega=-\infty}^{\infty}\delta(s-\omega)ds \\ &= \int_{-\infty}^{\infty} e^{jst}\text{III}(s)ds.\end{aligned}$$

Fourier Transform of the Shah Function

Substituting $2\pi\tau$ for t yields

$$\begin{aligned} \text{III}(\tau) &= \int_{-\infty}^{\infty} \text{III}(\text{s}) e^{j2\pi s\tau} ds \\ &= \mathcal{F}^{-1}\{\text{III}(s)\}(\tau). \end{aligned}$$

Consequently we see that

$$\mathcal{F}\{\text{III}\} = \text{III}.$$