Quadratic Formula

A *quadratic equation* is a second order polynomial:

$$y = ax^2 + bx + c.$$

If $a = 1$, $b = 0$, and $c = -2$, then

$$y = x^2 - 2.$$

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives solutions to the following equation:

$$ax^2 + bx + c = 0.$$

The solutions to $x^2 - 2 = 0$ are:

$$x = \frac{\pm\sqrt{8}}{2} = \frac{\pm 2\sqrt{2}}{2} = \pm\sqrt{2}.$$

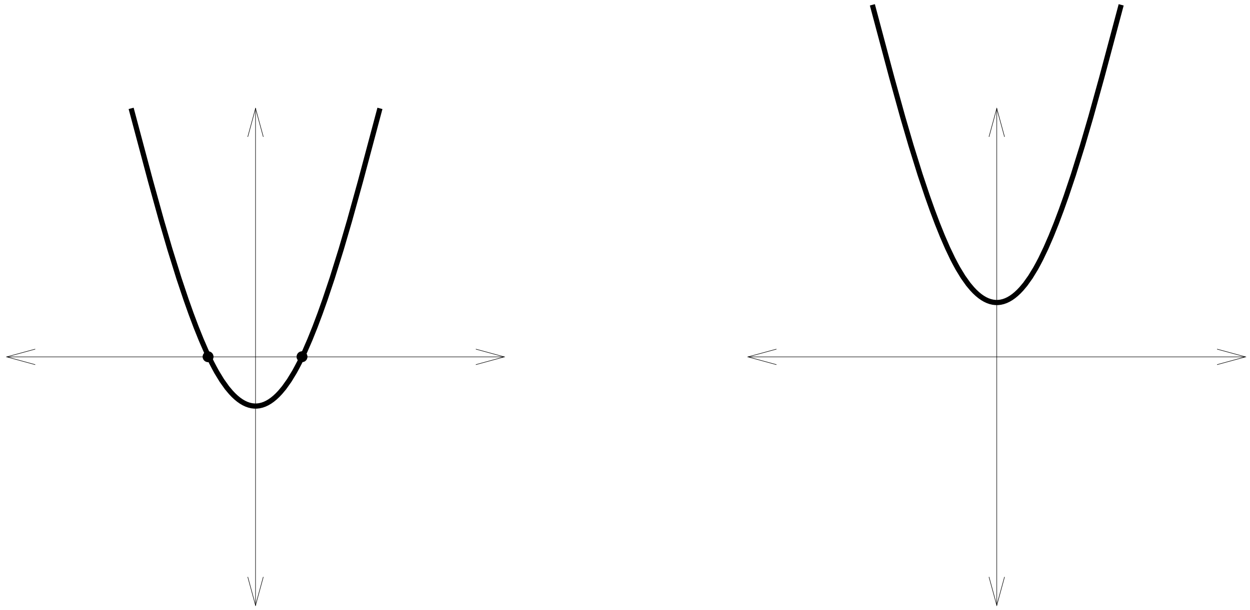

Figure 1: Left: $y = x^2 - 2$. Right: $y = x^2 + 2$.

The Square Root of -1

If $a = 1$, $b = 0$, and $c = 2$, then $y = x^2 + 2$. The quadratic formula gives solutions to $x^2 + 2 = 0$:

$$x = \frac{\pm\sqrt{-8}}{2} = \frac{\pm 2\sqrt{-2}}{2} = \pm\sqrt{-2}.$$

We can't take the square root of a negative number! So we use an *imaginary* number, i, to represent the $\sqrt{-1}$:

$$x = \pm\sqrt{-2} = \pm i\sqrt{2}.$$

The Square Root of -1 (contd).

Imaginary numbers can be added just like real numbers:

$$7i + 5i = 12i.$$

Can they be multiplied?

$$7i \cdot 5i = 35i^2 = 35\sqrt{-1}^2 = -35$$

which is real, not imaginary.

Complex Numbers

A complex number, c, has a real part, x, and an imaginary part, y:

$$c = x + yi.$$

A complex number can be decomposed into its real and imaginary parts:

$$\begin{aligned} \mathrm{Re}(c) &= x \\ \mathrm{Im}(c) &= y. \end{aligned}$$

Two complex numbers, c_1 and c_2, are equal, if and only if

$$\mathrm{Re}(c_1) = \mathrm{Re}(c_2)$$

and

$$\mathrm{Im}(c_1) = \mathrm{Im}(c_2).$$

In other words,

$$(x + iy) = (u + iv) \text{ iff } x = u \text{ and } y = v.$$

Adding Complex Numbers

Two complex numbers, c_1 and c_2, can be added:

$$\begin{aligned} \mathrm{Re}(c_1+c_2) &= \mathrm{Re}(c_1)+\mathrm{Re}(c_2) \\ \mathrm{Im}(c_1+c_2) &= \mathrm{Im}(c_1)+\mathrm{Im}(c_2). \end{aligned}$$

The real parts are added to form the real part of the sum and the imaginary parts are added to form the imaginary part of the sum. In other words, if $c_1 = x+yi$ and $c_2 = u+vi$, then

$$(x+yi)+(u+vi) = (x+u)+(y+v)i.$$

Euler's Equation

The Taylor series for $\sin x$, $\cos x$ and e^x are:

$$\begin{aligned}
\cos x &= \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots \\
\sin x &= \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots \\
e^x &= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\end{aligned}$$

Euler's Equation (contd.)

What is the Taylor series for e^{ix}?

$$
\begin{aligned}
e^{ix} &= \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots \\
&= \frac{i^0 x^0}{0!} + \frac{i^1 x^1}{1!} + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \ldots \\
&= \frac{x^0}{0!} + i\frac{x^1}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \ldots \\
&= \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) \\
&= \cos x + i \sin x.
\end{aligned}
$$

Polar to Rectangular

We have just derived *Euler's equation*:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

If we multiply both sides of the equation by a we get:

$$ae^{i\theta} = a\cos\theta + ia\sin\theta = x + iy$$

where a is *amplitude* and θ is *phase*. The left side of the equation is a complex number in *polar form*. The right side is a complex number in *rectangular form*:

$$\begin{aligned}\mathrm{Re}(ae^{i\theta}) &= a\cos\theta = x \\ \mathrm{Im}(ae^{i\theta}) &= a\sin\theta = y.\end{aligned}$$

The above equations show how we can convert from polar to rectangular. How do we go from rectangular to polar?

Rectangular to Polar

To solve for amplitude, a, given $x+iy$, we use the fact that $x = a\cos\theta$ and $y = a\sin\theta$ to write the following equation:

$$a^2\cos^2\theta + a^2\sin^2\theta = x^2 + y^2$$

which can be rearranged to yield

$$a^2\left(\cos^2\theta + \sin^2\theta\right) = x^2 + y^2$$

and since $\cos^2\theta + \sin^2\theta = 1$, it follows that

$$a = \sqrt{x^2 + y^2}.$$

To solve for phase, θ, we use the fact that $x = a\cos\theta$ and $y = a\sin\theta$ to write the following equation:

$$\tan\theta = \frac{a\sin\theta}{a\cos\theta} = \frac{y}{x}$$

which can be directly solved for θ:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

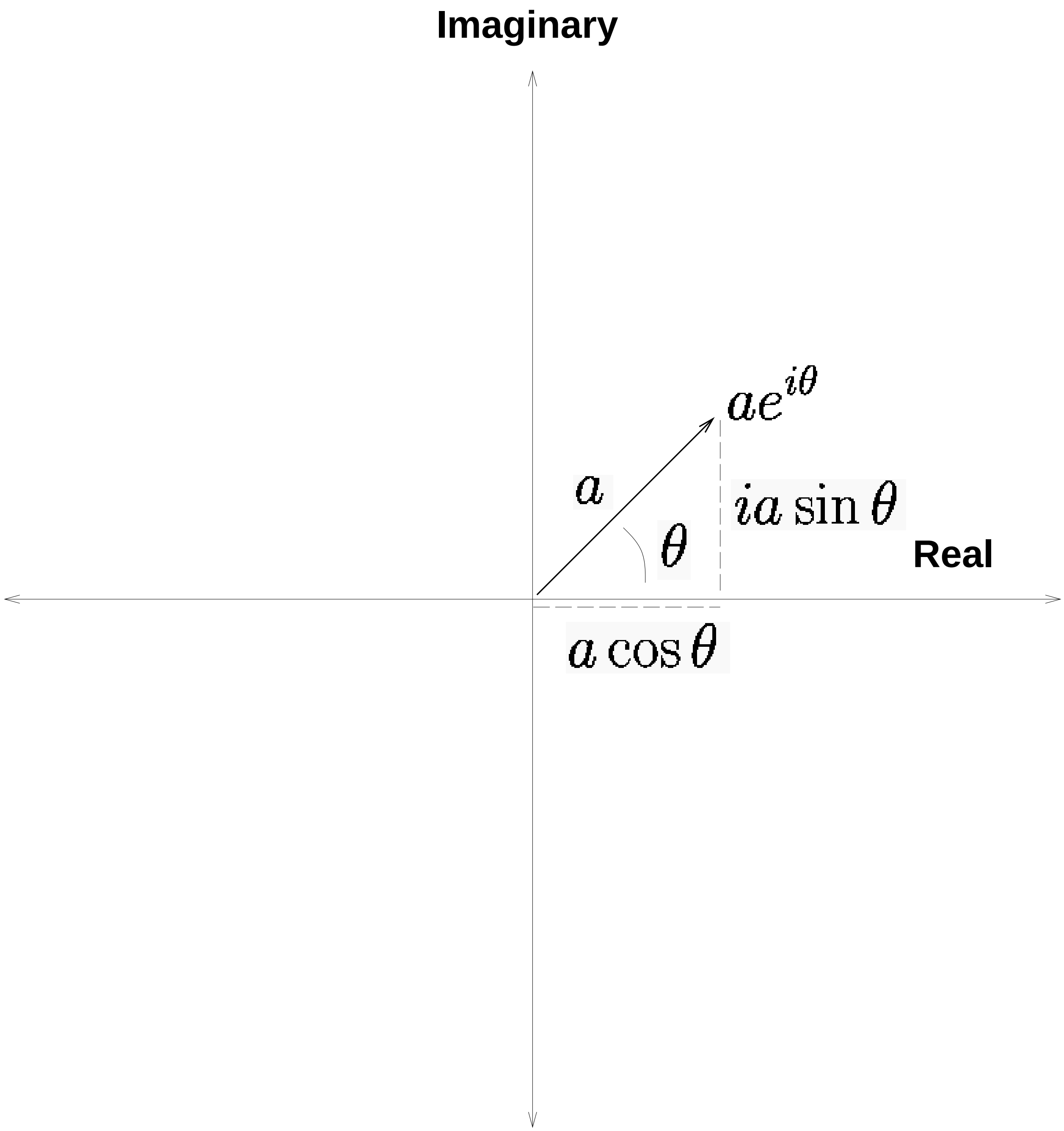

Figure 2: The complex plane.

Multiplication in Rectangular Form

Using the distributive property of multiplication, and the fact that $i = \sqrt{-1}$, we can derive the rules for multiplying two complex numbers in rectangular form:

$$\begin{aligned}(x+iy)\cdot(u+iv) &= xu+ixv+iyu-yv \\ &= (xu-yv)+i(xv+yu).\end{aligned}$$

To summarize, given two complex numbers, c_1 and c_2:

$$\begin{aligned}\mathrm{Re}(c_1 \cdot c_2) &= \mathrm{Re}(c_1)\mathrm{Re}(c_2) - \mathrm{Im}(c_1)\mathrm{Im}(c_2) \\ \mathrm{Im}(c_1 \cdot c_2) &= \mathrm{Re}(c_1)\mathrm{Im}(c_2) + \mathrm{Im}(c_1)\mathrm{Re}(c_2)\end{aligned}$$

...or rewriting this in matrix notation:

$$\begin{bmatrix}\mathrm{Re}(c_1 \cdot c_2) \\ \mathrm{Im}(c_1 \cdot c_2)\end{bmatrix} = \begin{bmatrix}\mathrm{Re}(\mathrm{c}_2) & -\mathrm{Im}(c_2) \\ \mathrm{Im}(\mathrm{c}_2) & \mathrm{Re}(c_2)\end{bmatrix}\begin{bmatrix}\mathrm{Re}(c_1) \\ \mathrm{Im}(c_1)\end{bmatrix}.$$

But this is pretty hard to remember.

Multiplication in Polar Form

Given two complex numbers in polar form:

$$
\begin{aligned}
c_1 &= a_1 e^{i\theta_1} \\
c_2 &= a_2 e^{i\theta_2}.
\end{aligned}
$$

We start with the rules for multiplying two complex numbers in rectangular form:

$$
\begin{bmatrix} \mathrm{Re}(c_1 \cdot c_2) \\ \mathrm{Im}(c_1 \cdot c_2) \end{bmatrix} = \begin{bmatrix} \mathrm{Re}(\mathrm{c}_2) & -\mathrm{Im}(c_2) \\ \mathrm{Im}(\mathrm{c}_2) & \mathrm{Re}(c_2) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(c_1) \\ \mathrm{Im}(c_1) \end{bmatrix}.
$$

Now substitute $a_2 \cos\theta_2$ for $\mathrm{Re}(c_2)$, $a_2 \sin\theta_2$ for $\mathrm{Im}(c_2)$, $a_1 \cos\theta_1$ for $\mathrm{Re}(\mathrm{c}_1)$, and $a_1 \sin\theta_1$ for $\mathrm{Im}(c_1)$ to get:

$$
\begin{bmatrix} \mathrm{Re}(c_1 \cdot c_2) \\ \mathrm{Im}(c_1 \cdot c_2) \end{bmatrix} = \begin{bmatrix} a_2 \cos\theta_2 & -a_2 \sin\theta_2 \\ a_2 \sin\theta_2 & a_2 \cos\theta_2 \end{bmatrix} \begin{bmatrix} a_1 \cos\theta_1 \\ a_1 \sin\theta_1 \end{bmatrix}.
$$

Multiplying this out yields:

$$
\begin{aligned}
\mathrm{Re}(c_1 \cdot c_2) &= a_1 a_2 \cos\theta_1 \cos\theta_2 - a_1 a_2 \sin\theta_1 \sin\theta_2 \\
\mathrm{Im}(c_1 \cdot c_2) &= a_1 a_2 \cos\theta_1 \sin\theta_2 + a_1 a_2 \sin\theta_1 \cos\theta_2.
\end{aligned}
$$

Multiplication in Polar Form (contd.)

Using the identities:

$$\begin{aligned}\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\\ \sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\end{aligned}$$

we see that

$$\begin{aligned}\mathrm{Re}(c_1\cdot c_2) &= a_1a_2\cos(\theta_1+\theta_2)\\ \mathrm{Im}(c_1\cdot c_2) &= a_1a_2\sin(\theta_1+\theta_2).\end{aligned}$$

It follows that

$$a_1e^{i\theta_1}\cdot a_2e^{i\theta_2} = a_1a_2e^{i(\theta_1+\theta_2)}.$$

To summarize, we multiply the amplitudes and sum the phases.

Observation

It is easy to add two complex numbers in rectangular form but hard in polar. Conversely it is easy to multiply two complex numbers in polar form but hard in rectangular.

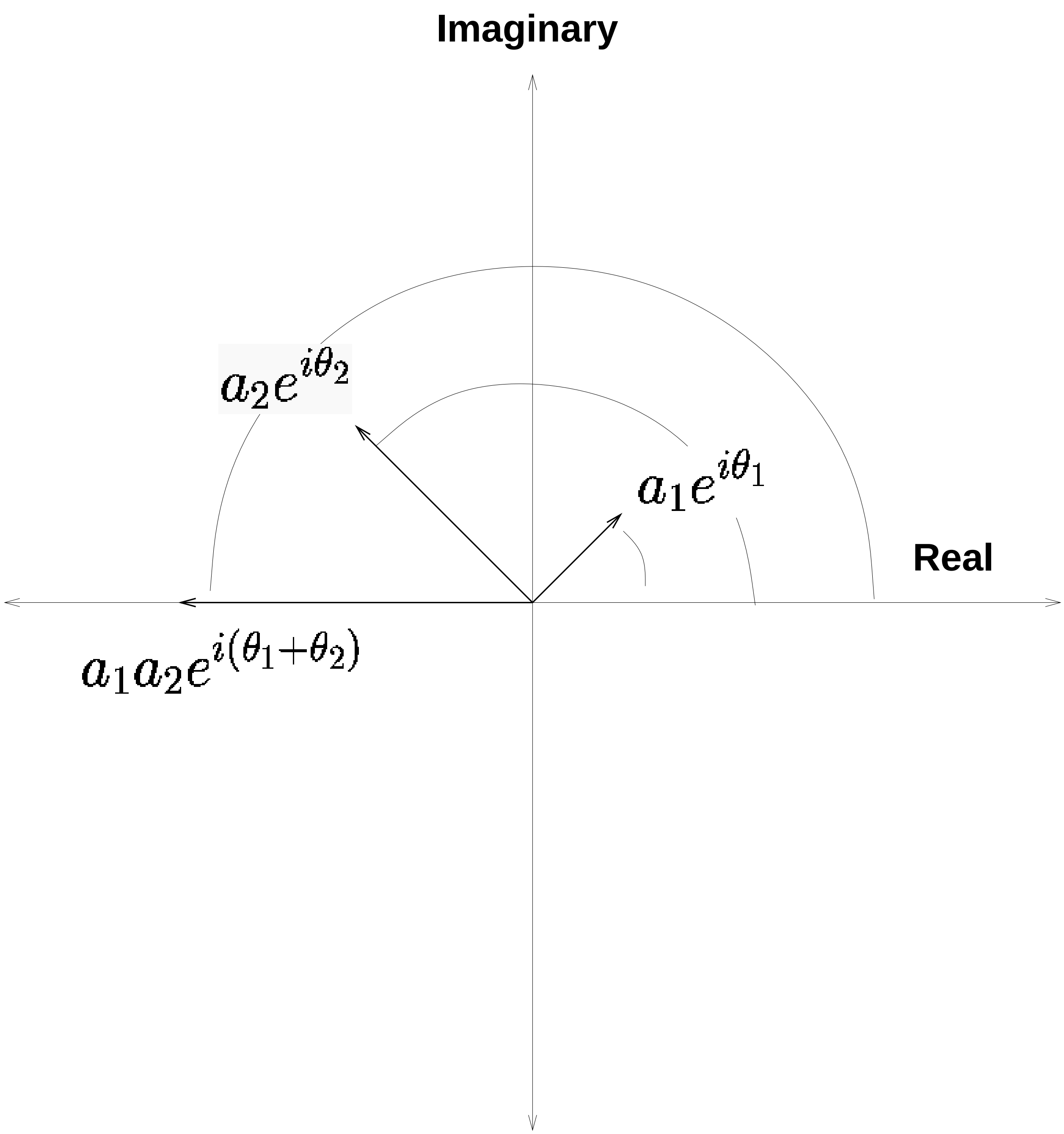

Figure 3: Multiplication of complex numbers.

Square root of a complex number

- A complex number always has two square roots.
- We find the first square root by taking the square root of the amplitude, a, and dividing the phase, θ, by 2.
- The second square root has the same amplitude, but is at $\theta/2+\pi$.
- Consequently, the square roots of $ae^{i\theta}$ are $\sqrt{a}e^{i\theta/2}$ and $\sqrt{a}e^{i(\theta/2+\pi)}$.

N-th roots of unity

- **Question** How many square roots does 1 have?
- **Answer** Two. They are both real.
- **Question** How many cube roots does 1 have?
- **Answer** Three. Only one of which is real.
- **Question** How many fourth roots does 1 have?
- **Answer** Four. Two are real. Two are imaginary.

Do you see a pattern?

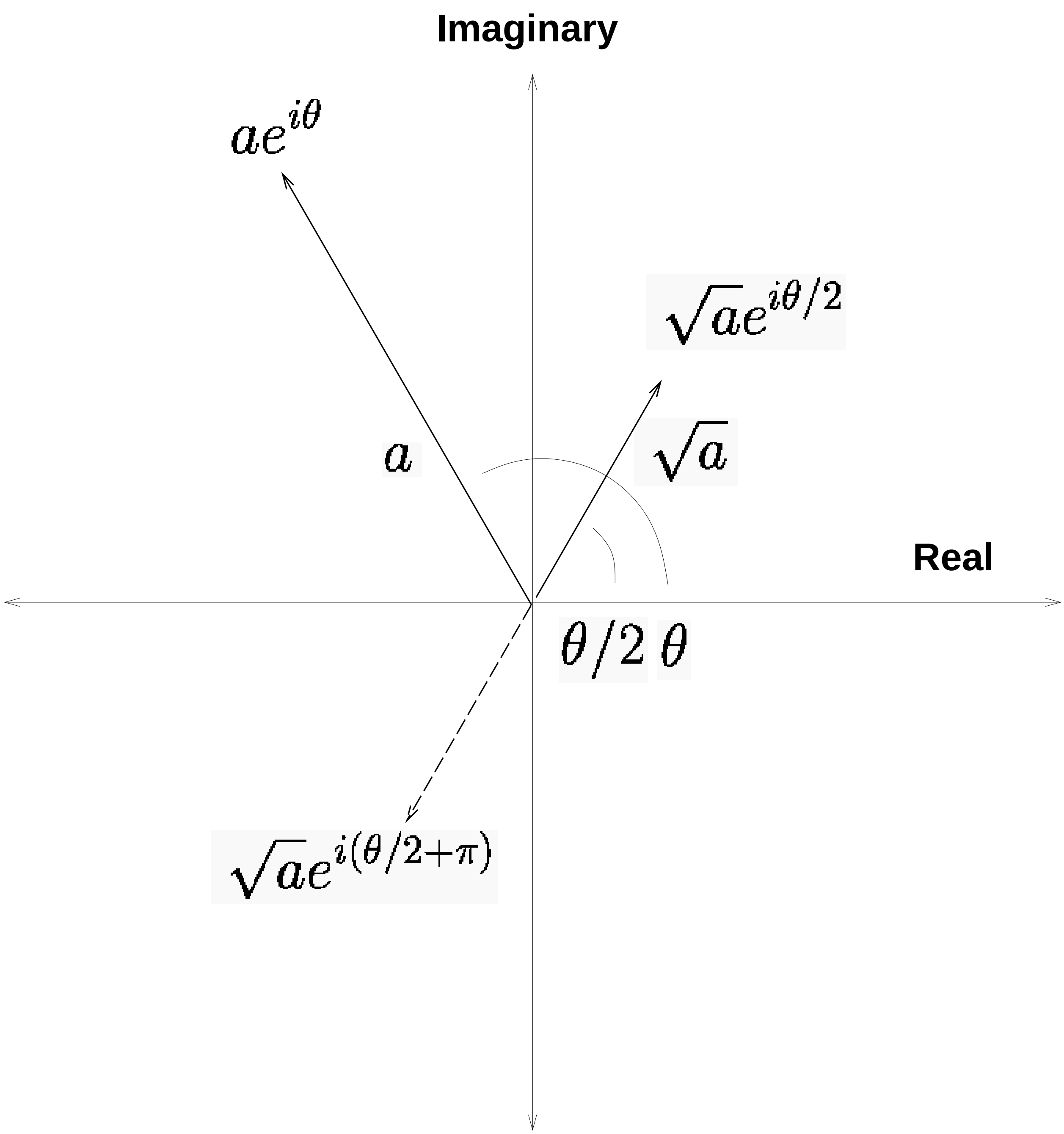

Figure 4: The square roots of $ae^{i}\theta$ are $\sqrt{a}e^{\theta/2}$ and $\sqrt{a}e^{(\theta/2+\pi)}$.

Complex conjugate

$$\begin{aligned} c &= x + iy \\ c^* &= x - iy \end{aligned}$$

...or, in polar form:

$$\begin{aligned} c &= ae^{i\theta} \\ c^* &= ae^{-i\theta} \end{aligned}$$

The complex conjugate has the same amplitude, but the phase is multiplied by minus one.

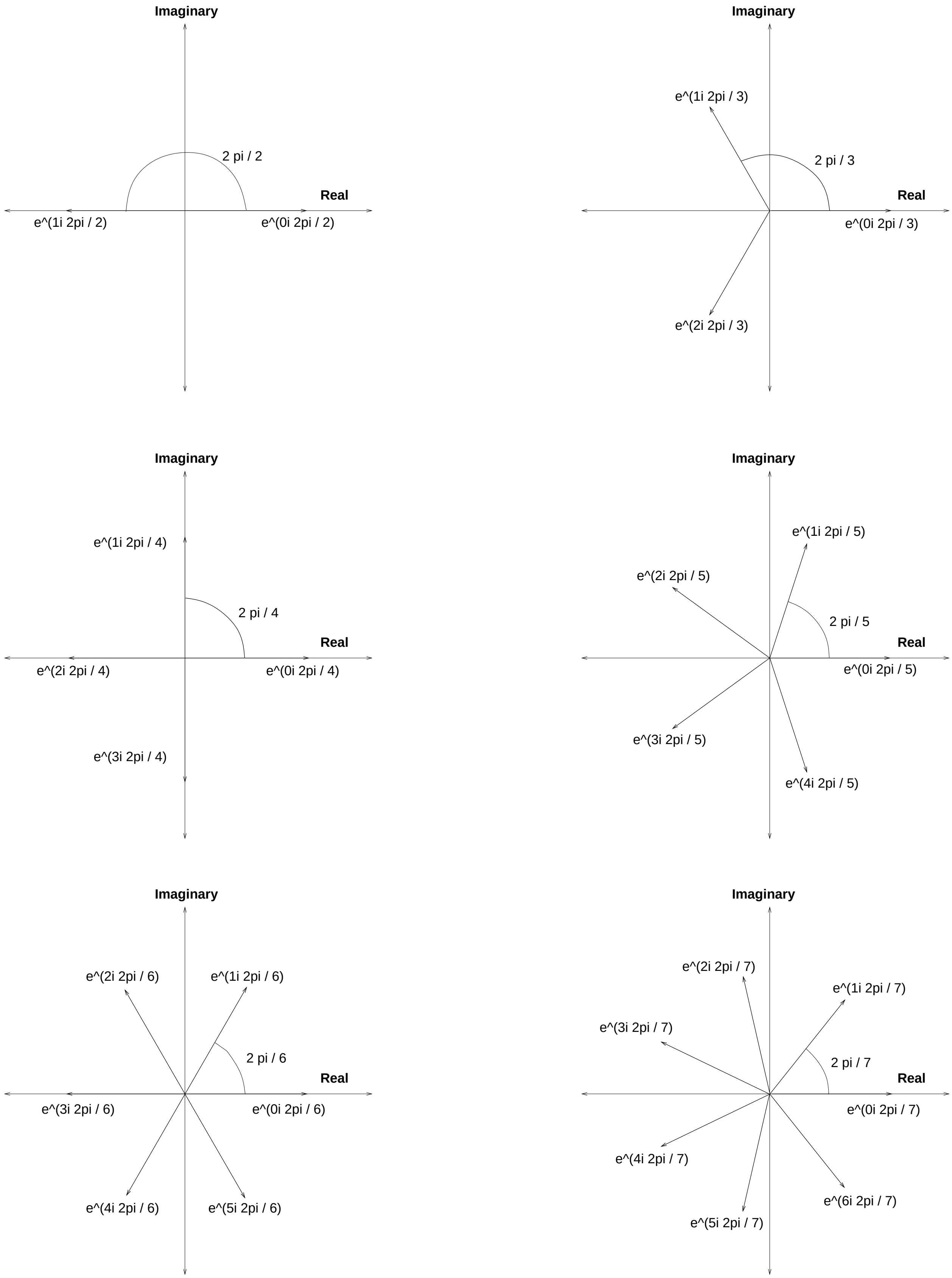

Figure 5: The N-th roots of unity.

Complex conjugate (contd.)

The sum of a conjugate pair:

$$\begin{aligned} c + c^* &= (x+iy) + (x-iy) \\ &= 2x \\ &= 2\text{Re}(c) \end{aligned}$$

The product of a conjugate pair:

$$\begin{aligned} cc^* &= (x+iy)(x-iy) \\ &= x^2 + iy - iy - i^2y^2 \\ &= x^2 + y^2 \\ &= a^2 \end{aligned}$$

...or, in polar form:

$$\begin{aligned} cc^* &= ae^{i\theta}ae^{-i\theta} \\ &= a^2e^{i(\theta-\theta)} \\ &= a^2 \end{aligned}$$

The amplitude of a complex number is the square root of the product of the complex number and its conjugate:

$$|c| = \sqrt{cc^*}.$$

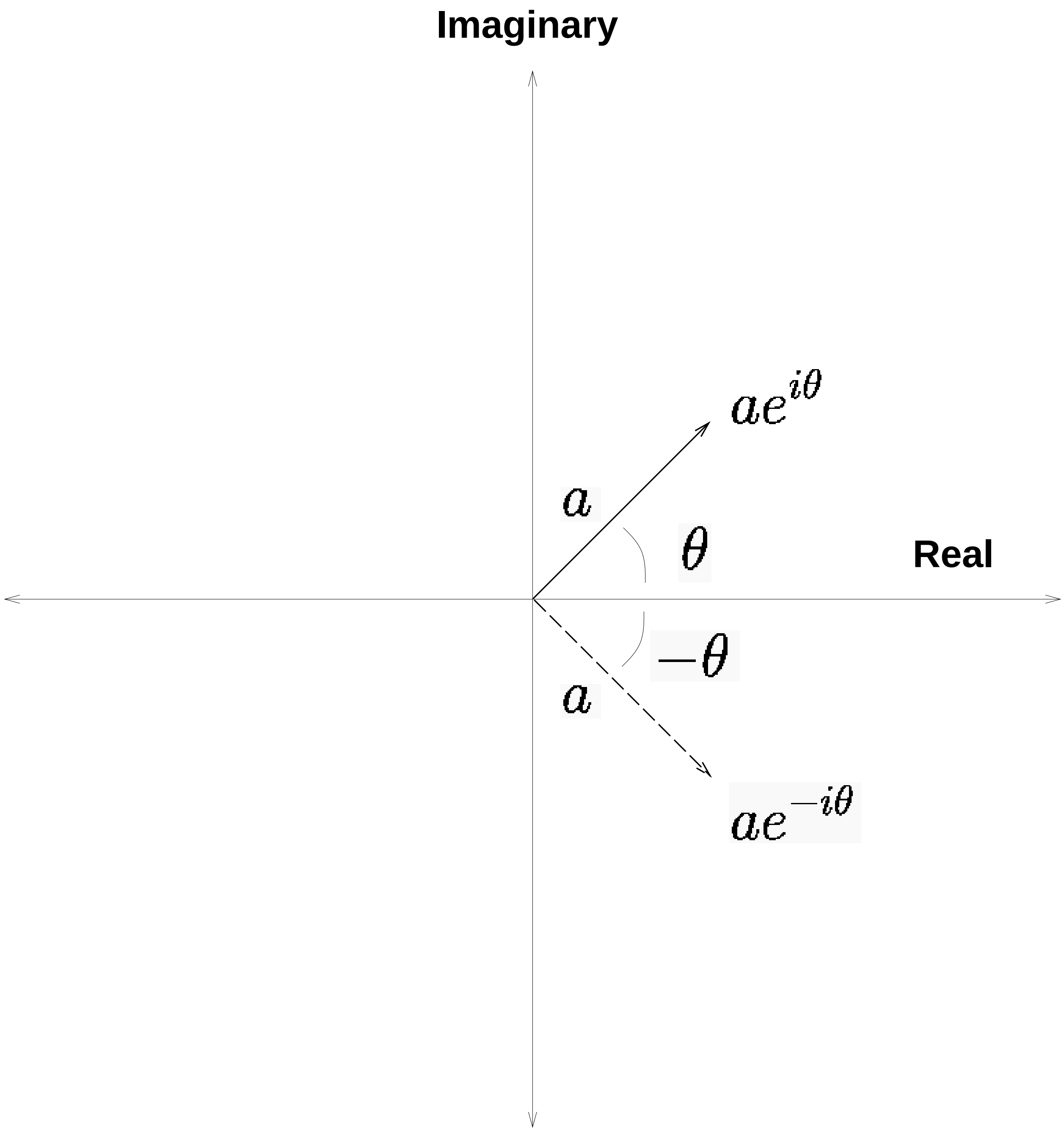

Figure 6: Complex conjugate.