Continuous Random Variables

The probability that a *continuous random variable*, X, has a value between a and b is computed by integrating its *probability density function (p.d.f.)* over the interval $[a,b]$:

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

A p.d.f. must integrate to one:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

Continuous Random Variables (contd.)

The probability that the continuous random variable, X, has any exact value, a, is 0:

$$\begin{aligned} P(X = a) &= \lim_{\Delta x \to 0} P(a \leq X \leq a + \Delta x) \\ &= \lim_{\Delta x \to 0} \int_a^{a+\Delta x} f_X(x) dx \\ &= 0. \end{aligned}$$

In general

$$P(X = a) \neq f_X(a).$$

Probability Density

The probability density at a multiplied by ε approximately equals the probability mass contained within an interval of ε width centered on a:

$$\begin{aligned}
\varepsilon f_X(a) &\approx \int_{a-\varepsilon/2}^{a+\varepsilon/2} f_X(x)dx \\
&\approx P(a-\varepsilon/2 \leq X \leq a+\varepsilon/2)
\end{aligned}$$

Cumulative Distribution Function

A continuous random variable, X, can also be defined by its *cumulative distribution function (c.d.f.)*:

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)dx.$$

For any c.d.f., $F_X(-\infty) = 0$ and $F_X(\infty) = 1$. The probability that a continuous random variable, X, has a value between a and b is easily computed using the c.d.f.:

$$\begin{aligned} P(a \leq X \leq b) &= \int_{a}^{b} f_X(x)dx \\ &= \int_{-\infty}^{b} f_X(x)dx - \int_{-\infty}^{a} f_X(x)dx \\ &= F_X(b) - F_X(a). \end{aligned}$$

Cumulative Distribution Function (contd.)

The p.d.f., $f_X(x)$, can be derived from the c.d.f., $F_X(x)$:

$$\begin{aligned} f_X(x) &= \frac{d}{dx}\int_{-\infty}^{x} f_X(s)ds \\ &= \frac{dF_X(x)}{dx}. \end{aligned}$$

Joint Probability Densities

Let X and Y be continuous random variables. The probability that $a \leq X \leq b$ **and** $c \leq Y \leq d$ is found by integrating the *joint probability density function* for X and Y over the interval $[a,b]$ w.r.t. x and over the interval $[c,d]$ w.r.t. y:

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{XY}(x,y)dydx.$$

Like a one-dimensional p.d.f., a two-dimensional joint p.d.f. must also integrate to one:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1.$$

Marginal Probability Densities

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy \\
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx
\end{aligned}
$$

Conditional Probability Densities

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\
&= \frac{f_{XY}(x,y)}{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}
\end{aligned}
$$

$$
f_{XY}(x,y) = f_{X|Y}(x|y) f_Y(y)
$$

Exponential Density

A constant fraction of a radioactive sample decays per unit time:

$$\frac{df(t)}{dt} = -\frac{1}{\tau} f(t).$$

What fraction of the radioactive sample will remain after time t?

$$\frac{d(e^{-\frac{t}{\tau}})}{dt} = -\frac{1}{\tau} e^{-\frac{t}{\tau}}$$

Exponential Density (contd.)

The function, $f(t) = e^{-\frac{t}{\tau}}$, satisfies the differential equation, but it does not integrate to one:

$$\begin{aligned} \int_0^\infty e^{-\frac{t}{\tau}} dt &= -\tau e^{-\frac{t}{\tau}} \Big|_0^\infty \\ &= \tau e^{-\frac{\infty}{\tau}} + \tau \\ &= \tau. \end{aligned}$$

So that $\int_{-\infty}^{\infty} f_T(t) dt = 1$, we divide $f(t)$ by τ:

$$f_T(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}.$$

Exponential Density (contd.)

The time, T, at which an atom of a radioactive element decays is a continuous random variable with the following p.d.f.:

$$f_T(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}.$$

The corresponding c.d.f. is:

$$\begin{aligned} F_T(a) &= \int_0^a \frac{1}{\tau} e^{-\frac{t}{\tau}} dt \\ &= -e^{-\frac{t}{\tau}} \Big|_0^a \\ &= 1 - e^{-\frac{a}{\tau}}. \end{aligned}$$

The c.d.f. gives the probability that an atom of a radioactive element has *already* decayed.

Example

The lifetime of a radioactive element is a continuous random variable with the following p.d.f.:

$$f_T(t) = \frac{1}{100}e^{-\frac{t}{100}}.$$

The probability that an atom of this element will decay within 50 years is:

$$\begin{aligned} P(0 \leq t \leq 50) &= \int_0^{50} \frac{1}{100}e^{-\frac{t}{100}}dt \\ &= 1 - e^{-0.5} \\ &= 0.39. \end{aligned}$$

Exponential Density (contd.)

The *half-life*, λ, is defined as the time required for half of a radioactive sample to decay:

$$P(0 \leq t \leq \lambda) = 1/2.$$

Since

$$\begin{aligned} P(0 \leq t \leq \lambda) &= \int_0^{\lambda} \frac{1}{100} e^{-\frac{t}{100}} dt \\ &= 1 - e^{-\frac{1}{100}\lambda} \\ &= 1/2, \end{aligned}$$

it follows that $\lambda = 100 \ln 2$ or 69.31 years.

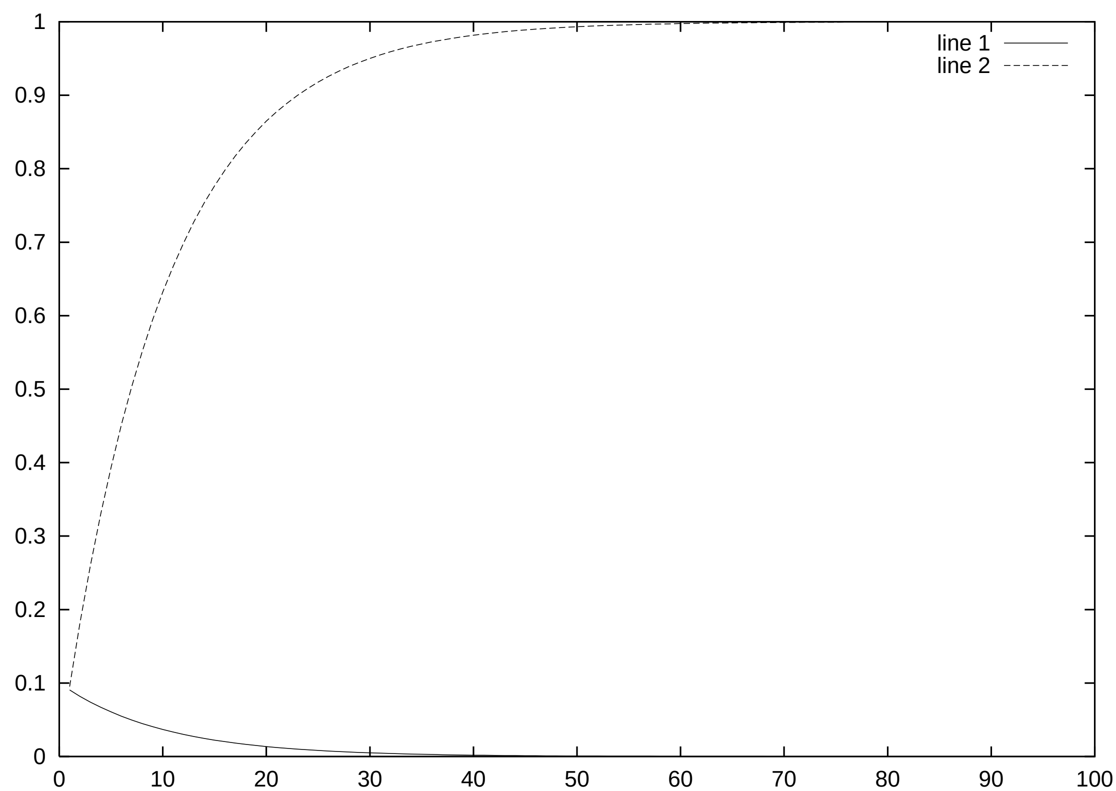

Figure 1: Exponential p.d.f., $\frac{1}{10}e^{-\frac{1}{10}t}$, and c.d.f., $1-e^{-\frac{1}{10}t}$.

Memoryless Property of the Exponential

If X is an exponentially distributed random variable, then

$$P(X > s+t | X > t) = P(X > s).$$

Proof:

$$\begin{aligned} P(X > s+t | X > t) &= \frac{P(X > s+t, X > t)}{P(X > t)} \\ &= \frac{P(X > t | X > s+t) P(x > s+t)}{P(X > t)} \\ &= \frac{P(X > s+t)}{P(X > t)}. \end{aligned}$$

Since $P(X > t) = 1 - P(X \leq t)$,

$$\begin{aligned} \frac{P(X > s+t)}{P(X > t)} &= \frac{1 - (1 - e^{-(s+t)/\tau})}{1 - (1 - e^{-t/\tau})} \\ &= e^{-s/\tau} \\ &= P(X > s). \end{aligned}$$

Memoryless Property of the Exponential

In plain language: Knowing how long we've already waited doesn't tells us anything about how much longer we are going to have to wait, *e.g.*, for a bus.

Expected Value

Let X be a continuous random variable. The *expected value* of X, is defined as follows:

$$\langle X \rangle = \mu = \int_{-\infty}^{\infty} x f_X(x) dx$$

Variance

The *variance* of X is defined as the expected value of the squared difference of X and $\langle X \rangle$:

$$\left\langle [X - \langle X \rangle]^2 \right\rangle = \sigma^2 = \int_{-\infty}^{\infty} [x - \langle X \rangle]^2 f_X(x) dx$$

Gaussian Density

A random variable X with p.d.f.,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

is called a *Gaussian* (or *normal*) random variable with *expected value*, μ, and *variance*, σ^2.

Expected Value for Gaussian Density

Let the p.d.f., $f_X(X)$, equal

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}.$$

The expected value, $\langle X \rangle$, can be derived as follows:

$$\langle X \rangle = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)} dx.$$

Expected Value for Gaussian Density (contd.)

Writing x as $(x-\mu)+\mu$:

$$\begin{aligned}\langle X \rangle &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}dx \\ &\quad + \mu\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}e^{-(x-\mu)^2/(2\sigma^2)}dx.\end{aligned}$$

The first term is zero, since it is the integral of the product of an odd and even function. The second term is μ, since

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}e^{-(x-\mu)^2/(2\sigma^2)}dx = \int_{-\infty}^{\infty}f_X(x)dx = 1.$$

Consequently,

$$\langle X \rangle = \mu.$$

C.d.f. for Gaussian Density

Because the Gaussian integrates to one and is symmetric about zero, its c.d.f., $F_X(a)$, can be written as follows:

$$\int_{-\infty}^{a} f_X(x)dx = \begin{cases} \frac{1}{2} - \int_a^0 f_X(x)dx & \text{if } a < 0 \\ \frac{1}{2} + \int_0^a f_X(x)dx & \text{otherwise.} \end{cases}$$

Equivalently, we can write:

$$\int_{-\infty}^{a} f_X(x)dx = \begin{cases} \frac{1}{2} - \int_0^{|a|} f_X(x)dx & \text{if } a < 0 \\ \frac{1}{2} + \int_0^{|a|} f_X(x)dx & \text{otherwise.} \end{cases}$$

C.d.f. for Gaussian Density (contd).

To evaluate $\int_0^{|a|} f_X(x)dx$, recall that the Taylor series for e^x is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The Taylor series for a Gaussian is therefore:

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\left(-x^2/2\right)^n}{n!} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n}}{2^n}. \end{aligned}$$

C.d.f. for Gaussian Density (contd.)

Consequently:

$$\begin{aligned}
\int_0^{|a|} f_X(x)dx &= \frac{1}{\sqrt{2\pi}} \int_0^{|a|} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n}}{2^n} dx \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2^n(2n+1)} \Bigg|_0^{|a|} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{|a|^{2n+1}}{2^n(2n+1)}.
\end{aligned}$$

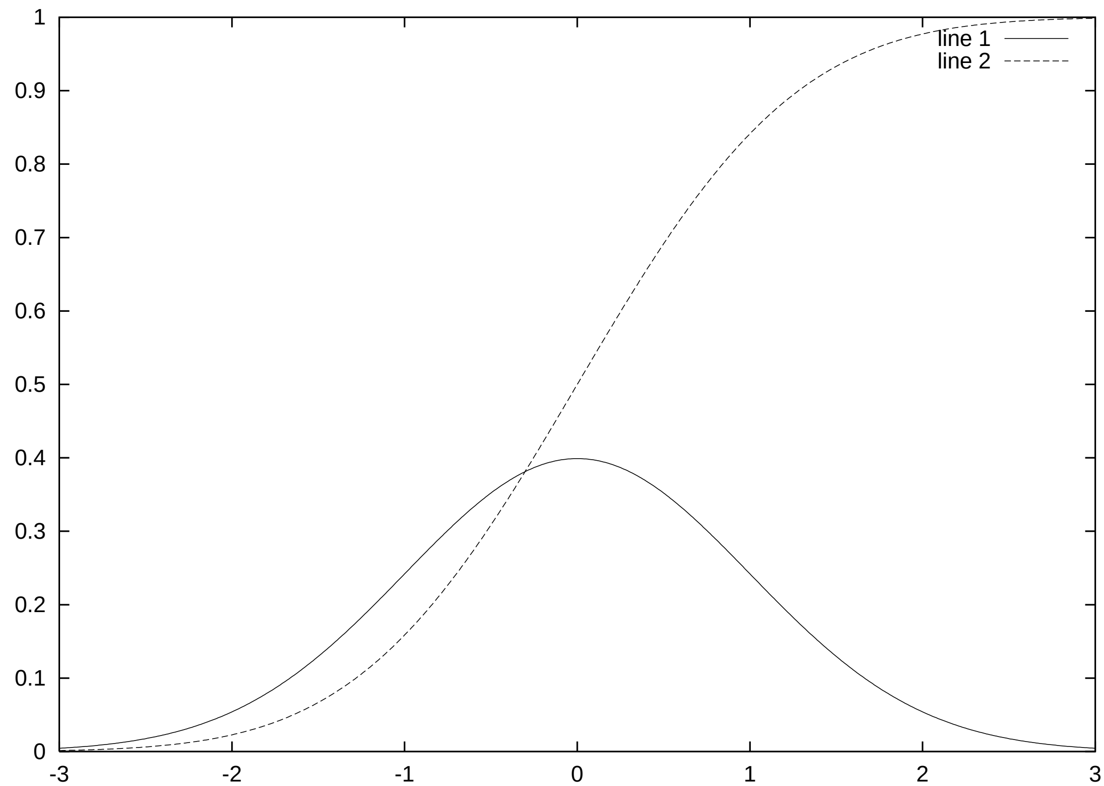

Figure 2: Gaussian p.d.f. and c.d.f., $\mu = 0$ and $\sigma^2 = 1$, computed using Taylor series.