The Discrete Fourier Transform (DFT)

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Sampling Periodic Functions

Given a function of period, T, *i.e.*,

$$f(t) = f(t+T)$$

choose N and **sample** $f(t)$ within the interval, $0 \leq t \leq T$, at N equally spaced points, $n\Delta t$, where $n = 0, 1, \ldots, N-1$ and $\Delta t = T/N$. The result is a discrete function of period, N, which can be represented as a vector, $\mathbf{f}$, in $\mathbb{R}^N$ (or $\mathbb{C}^N$) where $f_n = f(n\Delta t)$:

$$\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}.$$

Inner Product of Discrete Periodic Functions

We can define the *inner product* of two discrete functions of period, N, as follows:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{n=0}^{N-1} f_n^* \, g_n.$$

Standard Basis

$$(\mathbf{k}_m)_n = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

Example:

$$\mathbf{k}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Because $\langle \mathbf{k}_{m_1}, \mathbf{k}_{m_2} \rangle$ equals zero when $m_1 \neq m_2$ and one when $m_1 = m_2$, the set of $\mathbf{k}_m$ for $0 \leq m < N$ form an orthonormal basis for $\mathbb{R}^N$ (or $\mathbb{C}^N$) and therefore for discrete functions of period, N.

Sampled Harmonic Signal Basis

A sampled harmonic signal is a discrete function of period, N:

$$W_{n,m} = \frac{1}{\sqrt{N}} e^{\,j2\pi m \frac{n}{N}}$$

where m is frequency and n is position. A sampled harmonic signal of frequency, m, can be represented by a vector of length N:

$$\mathbf{w}_m = \begin{bmatrix} W_{0,m} \\ W_{1,m} \\ \vdots \\ W_{N-1,m} \end{bmatrix} = \frac{1}{\sqrt{N}} \begin{bmatrix} e^{\,j2\pi m \frac{0}{N}} \\ e^{\,j2\pi m \frac{1}{N}} \\ \vdots \\ e^{\,j2\pi m \frac{(N-1)}{N}} \end{bmatrix}.$$

Sampled Harmonic Signal Basis (contd.)

How "long" is a sampled harmonic signal?

$$
\begin{aligned}
\|\mathbf{w}_m\| &= \langle \mathbf{w}_m, \mathbf{w}_m \rangle^{\frac{1}{2}} \\
&= \left(\sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} e^{-j2\pi m \frac{n}{N}} \frac{1}{\sqrt{N}} e^{\,j2\pi m \frac{n}{N}} \right)^{\frac{1}{2}} \\
&= \left(\sum_{n=0}^{N-1} \frac{1}{N} \right)^{\frac{1}{2}} \\
&= 1
\end{aligned}
$$

Sampled Harmonic Signal Basis (contd.)

What is the "angle" between two sampled harmonic signals, $\mathbf{w}_{m_1}$ and $\mathbf{w}_{m_2}$, when $m_1 \neq m_2$?

$$\begin{aligned}
\langle \mathbf{w}_{m_1}, \mathbf{w}_{m_2} \rangle &= \frac{1}{N} \sum_{n=0}^{N-1} e^{-j2\pi m_1 \frac{n}{N}} e^{\,j2\pi m_2 \frac{n}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} e^{\,j2\pi (m_2 - m_1) \frac{n}{N}} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \left(e^{\,j2\pi \frac{(m_2 - m_1)}{N}} \right)^n
\end{aligned}$$

Sampled Harmonic Signal Basis (contd.)

Substituting α for $e^{j2\pi\frac{(m_2-m_1)}{N}}$ yields

$$\langle \mathbf{w}_{m_1}, \mathbf{w}_{m_2} \rangle = \frac{1}{N} \sum_{n=0}^{N-1} \alpha^n$$

afterwhich the following identity:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

can be applied to yield

$$\langle \mathbf{w}_{m_1}, \mathbf{w}_{m_2} \rangle = \frac{1}{N} \left(\frac{1-\alpha^N}{1-\alpha} \right).$$

Sampled Harmonic Signal Basis (contd.)

Since $\alpha = e^{j2\pi\frac{(m_2-m_1)}{N}}$, it follows that

$$\begin{aligned} \alpha^N &= e^{j2\pi(m_2-m_1)\frac{N}{N}} \\ &= e^{j2\pi(m_2-m_1)}. \end{aligned}$$

Because $e^{j2\pi k} = 1$ for all integers, $k \neq 0$, and because $(m_2 - m_1) \neq 0$ is an integer, it follows that $\alpha^N = 1$ yet $\alpha \neq 1$. Consequently,

$$\begin{aligned} \langle \mathbf{w}_{m_1}, \mathbf{w}_{m_2} \rangle &= \frac{1}{N}\left(\frac{1-\alpha^N}{1-\alpha}\right) \\ &= 0. \end{aligned}$$

In summary, because $\langle \mathbf{w}_{m_1}, \mathbf{w}_{m_2} \rangle = 0$ when $m_1 \neq m_2$ and $\langle \mathbf{w}_{m_1}, \mathbf{w}_{m_2} \rangle = 1$ when $m_1 = m_2$, the set of $\mathbf{w}_m$ for $0 \leq m < N$ form an orthonormal basis for $\mathbb{R}^N$ (or $\mathbb{C}^N$) and therefore for discrete functions of period, N.

The Discrete Fourier Transform (DFT)

- **Question** What are the coefficients of $\mathbf{f}$ in the sampled harmonic signal basis?
- **Answer** Take inner products of $\mathbf{f}$ with the finite set of sampled harmonic signals, $\mathbf{w}_m$, for $0 \leq m < N$.

The result is the *analysis formula* for the DFT:

$$\begin{aligned} \hat{f}_m &= \langle \mathbf{w}_m, \mathbf{f} \rangle \\ &= \langle \frac{1}{\sqrt{N}} e^{j2\pi m \frac{n}{N}}, \mathbf{f} \rangle \\ &= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-j2\pi m \frac{n}{N}} \end{aligned}$$

where $\hat{\mathbf{f}}$ is used to denote the discrete Fourier transform of $\mathbf{f}$. The function can be reconstructed using the *synthesis formula* for the DFT:

$$f_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \hat{f}_m e^{j2\pi m \frac{n}{N}}.$$

The DFT in Matrix Form

The analysis formula for the DFT:

$$\hat{f}_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f_n e^{-j2\pi m \frac{n}{N}}$$

can be written as a matrix equation:

$$\begin{bmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix} = \begin{bmatrix} W^*_{0,0} & \cdots & W^*_{0,N-1} \\ \vdots & \ddots & \vdots \\ W^*_{N-1,0} & \cdots & W^*_{N-1,N-1} \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

where $W^*_{m,n} = \frac{1}{\sqrt{N}} e^{-j2\pi m \frac{n}{N}}$.

More concisely:

$$\hat{\mathbf{f}} = \mathbf{W}^* \mathbf{f}.$$

The DFT in Matrix Form (contd.)

The synthesis formula for the DFT:

$$f_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \hat{f}_m e^{\,j2\pi m \frac{n}{N}}$$

can also be written as a matrix equation:

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = \begin{bmatrix} W_{0,0} & \cdots & W_{N-1,0} \\ \vdots & \ddots & \vdots \\ W_{N-1,0} & \cdots & W_{N-1,N-1} \end{bmatrix} \begin{bmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix}$$

where $W_{m,n} = \frac{1}{\sqrt{N}} e^{\,j2\pi m \frac{n}{N}}$. More concisely:

$$\mathbf{f} = \mathbf{W}\hat{\mathbf{f}}.$$

Note: Because only the **product** of frequency, m, and position, n, appears in the expression for a sampled harmonic signal, it follows that $W_{m,n} = W_{n,m}$. Therefore $\mathbf{W} = \mathbf{W}^{\mathrm{T}}$. The only difference between the matrices used for the forward and inverse DFT's, *i.e.*, $\mathbf{W}^*$ and $\mathbf{W}$, is conjugation.

The DFT in Matrix Form (contd.)

A matrix product, $\mathbf{y} = \mathbf{A}\mathbf{x}$, can be interpreted in two different ways.

1. The i-th component of $\mathbf{y}$ is the inner product of $\mathbf{x}$ with the i-th row of $\mathbf{A}$:

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{0,0} & \dots & A_{0,N-1} \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} A_{N-1,0} & \dots & A_{N-1,N-1} \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_{N-1} \end{bmatrix} \end{bmatrix}$$

2. The vector, $\mathbf{y}$, is a linear combination of the columns of $\mathbf{A}$. The i-th column is weighted by x_i:

$$\begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix} = x_0 \begin{bmatrix} A_{0,0} \\ \vdots \\ A_{N-1,0} \end{bmatrix} + \cdots + x_{N-1} \begin{bmatrix} A_{0,N-1} \\ \vdots \\ A_{N-1,N-1} \end{bmatrix}$$

The DFT in Matrix Form (contd.)

Both ways of looking at matrix product are equally correct. However, it is useful to think of the analysis formula, $\hat{\mathbf{f}} = \mathbf{W}^*\mathbf{f}$, the first way:

$$\begin{bmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{N-1} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} W^*_{0,0} & \cdots & W^*_{0,N-1} \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} W^*_{N-1,0} & \cdots & W^*_{N-1,N-1} \end{bmatrix} \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} \end{bmatrix}$$

i.e., $\hat{f}_m$ is the inner product of $\mathbf{f}$ with the m-th row of $\mathbf{W}$. Conversely, it is useful to think of the synthesis formula, $\mathbf{f} = \mathbf{W}\hat{\mathbf{f}}$, the second way:

$$\begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} = \hat{f}_0 \begin{bmatrix} W_{0,0} \\ \vdots \\ W_{N-1,0} \end{bmatrix} + \cdots + \hat{f}_{N-1} \begin{bmatrix} W_{0,N-1} \\ \vdots \\ W_{N-1,N-1} \end{bmatrix}$$

i.e., $\mathbf{f}$ is a linear combination of the columns of $\mathbf{W}$. The m-th column is weighted by $\hat{f}_m$.

Convolution of Discrete Periodic Functions

Let $\mathbf{f}$ and $\mathbf{g}$ be vectors in $\mathbb{R}^N$. Because $\mathbf{f}$ and $\mathbf{g}$ represent discrete functions of period, N, we adopt the convention that $f(k \pm N) = f(k)$. The k-th component of the *convolution* of $\mathbf{f}$ and $\mathbf{g}$ is then

$$\{\mathbf{f} * \mathbf{g}\}_k = \sum_{j=0}^{N-1} f_j \, g_{k-j}.$$

Example of Discrete Periodic Convolution

Calculate $\{\mathbf{f} * \mathbf{g}\}_k$ when

$$\mathbf{g} = \begin{bmatrix} 2 & 1 & 0 & \ldots & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

Since $\mathbf{f} * \mathbf{g} = \mathbf{g} * \mathbf{f}$ and since

$$\{\mathbf{g} * \mathbf{f}\}_k = \sum_{j=0}^{N-1} g_j f_{k-j}$$

it follows that

$$\begin{aligned} \{\mathbf{f} * \mathbf{g}\}_k &= g_0 f_k + g_1 f_{k-1} + \cdots + g_{N-1} f_{k-(N-1)} \\ &= 2 f_k + 1 f_{k-1} + 1 f_{k-(N-1)} \\ &= f_{k-1} + 2 f_k + 1 f_{k+1} \end{aligned}$$

This operation performs a local weighted averaging of $\mathbf{f}$.

Circulant Matrices

The convolution formula for discrete periodic functions

$$\{\mathbf{f} * \mathbf{g}\}_k = \sum_{j=0}^{N-1} f_j g_{k-j}$$

can be written as a matrix equation:

$$\mathbf{f} * \mathbf{g} = \mathbf{C}\mathbf{f}$$

where $C_{k,j} = g_{k-j}$.

$$\mathbf{C} = \begin{bmatrix} g_0 & g_{N-1} & g_{N-2} & \cdots & g_1 \\ g_1 & g_0 & g_{N-1} & \cdots & g_2 \\ g_2 & g_1 & g_0 & \cdots & g_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{N-1} & g_{N-2} & g_{N-3} & \cdots & g_0 \end{bmatrix}$$

Matrices like **C** are termed *circulant*.

Matrix Diagonalization

A vector, $\mathbf{x}$, is a **right** eigenvector when $\mathbf{Ax}$ points in the same direction as $\mathbf{x}$ but is (possibly) of different length:

$$\lambda\mathbf{x} = \mathbf{Ax}$$

A vector, $\mathbf{y}$, is a **left** eigenvector when $\mathbf{y}^{\mathrm{T}}\mathbf{A}$ points in the same direction as $\mathbf{y}^{\mathrm{T}}$ but is (possibly) of different length:

$$\lambda\mathbf{y}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}\mathbf{A}$$

A diagonalizable matrix of rank, N, has N linearly independent right eigenvectors

$$\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}$$

and N linearly independent left eigenvectors

$$\mathbf{y}_0, \ldots, \mathbf{y}_{N-1}$$

which share the N eigenvalues

$$\lambda_0, \ldots, \lambda_{N-1}.$$

Matrix Diagonalization (contd.)

Such a matrix can be factored as follows:

$$\mathbf{A} = \mathbf{X}\Lambda\mathbf{Y}^{\mathrm{T}}$$

where the i-th column of $\mathbf{X}$ is $\mathbf{x}_i$ and the i-th row of $\mathbf{Y}^{\mathrm{T}}$ is $\mathbf{y}_i$ and Λ is diagonal with $\Lambda_{i,i} = \lambda_i$:

$$\Lambda = \begin{bmatrix} \lambda_0 & 0 & \ldots & 0 \\ 0 & \lambda_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{N-1} \end{bmatrix}$$

We also observe that

$$\mathbf{X}\mathbf{Y}^{\mathrm{T}} = \mathbf{I}$$

i.e., $\mathbf{X}$ and $\mathbf{Y}^{\mathrm{T}}$ are inverses. We say that $\mathbf{A}$ has been *diagonalized*. Stated differently, in the basis formed by its right eigenvectors, the linear operator, $\mathbf{A}$, is represented by the diagonal matrix, Λ.

Diagonalization of Circulant Matrices

When $\mathbf{C}$ is circulant the left and right eigenvectors are sampled *harmonic signals* and *conjugate harmonic signals*. Consequently, $\mathbf{X} = \mathbf{W}$ and $\mathbf{Y}^{\mathrm{T}} = \mathbf{W}^*$, and $\mathbf{C}$ can be factored as follows:

$$\mathbf{C} = \mathbf{W}\Lambda\mathbf{W}^*$$

where $W_{m,n} = \frac{1}{\sqrt{N}} e^{\,j2\pi m \frac{n}{N}}$ and

$$\Lambda = \begin{bmatrix} \hat{g}_0 & 0 & \dots & 0 \\ 0 & \hat{g}_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{g}_{N-1} \end{bmatrix}$$

where $\Lambda_{m,m} = \lambda_m = \hat{g}_m$, the m-th coefficient of the DFT of $\mathbf{g}$, the first column of $\mathbf{C}$.

Convolution Theorem

It follows that we can use the DFT to compute $\mathbf{f} * \mathbf{g}$:

$$\begin{array}{ccc} \hat{\mathbf{f}} & \stackrel{\Lambda}{\longrightarrow} & \Lambda\hat{\mathbf{f}} \\ \uparrow \mathbf{W}^* & & \downarrow \mathbf{W} \\ \mathbf{f} & \stackrel{\mathbf{C}}{\longrightarrow} & \mathbf{f} * \mathbf{g} \end{array}$$

In plain English, multiplication with a circulant matrix, $\mathbf{C}$, in the time domain is equivalent to multiplication with a diagonal matrix, Λ, in the frequency domain:

$$\mathbf{C}\mathbf{f} = \mathbf{W}\Lambda\mathbf{W}^*\mathbf{f}.$$

Polynomial Multiplication

$$p(x) = p_0 x^0 + p_1 x^1 + p_2 x^2 + \cdots + p_m x^m$$
$$q(x) = q_0 x^0 + q_1 x^1 + q_2 x^2 + \cdots + q_n x^n$$

$$p(x)q(x) = p_0 q_0 x^0 +$$
$$(p_0 q_1 + p_1 q_0) x^1 +$$
$$(p_0 q_2 + p_1 q_1 + p_2 q_0) x^2 +$$
$$(p_0 q_3 + p_1 q_2 + p_2 q_1 + p_3 q_0) x^3 +$$
$$(p_0 q_4 + p_1 q_3 + p_2 q_2 + p_3 q_1 + p_4 q_0) x^4 +$$
$$\vdots$$
$$(p_0 q_{n+m} + p_1 q_{n+m-1} + \cdots + p_{n+m-1} q_1 + p_{n+m} q_0) x^{n+m}$$

Polynomial Multiplication (contd.)

$$\begin{aligned} r(x) &= p(x)q(x) \\ &= r_0 x_0 + r_1 x_1 + r_2 x_2 + \cdots + r_{m+n} x^{m+n} \end{aligned}$$

where

$$\begin{aligned} r_i &= p_0 q_i + p_1 q_{i-1} + \cdots + p_{i-1} q_1 + p_i q_0 \\ &= \sum_{j=0}^{i} p_j q_{i-j} \\ &= \sum_{j=-\infty}^{\infty} p_j q_{i-j} \\ &= \{\mathbf{p} * \mathbf{q}\}_i \end{aligned}$$