Histogram Equalization

Last time we derived an expression for f_Y in terms of g', g^{-1} and f_X when $Y = g(X)$:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}.$$

A *uniform random variable*, U, has the following p.d.f.:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Does there exist a g, which will transform a random variable, X, with p.d.f., f_X, into a uniform random variable, U, with p.d.f., f_U?

Histogram Equalization (contd.)

We start by setting the expression for f_Y equal to the expression for f_U:

$$\frac{f_X(g^{-1}(u))}{g'(g^{-1}(u))} = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

and multiplying both sides by $f_X(g^{-1}(u))$

$$g'(g^{-1}(u)) = f_X(g^{-1}(u)) \text{ if } 0 < u < 1.$$

Histogram Equalization (contd.)

After substituting v for $g^{-1}(u)$ and $g(v)$ for u we have

$$g'(v) = f_X(v) \text{ if } 0 < g(v) < 1.$$

Integrating both sides of the equation:

$$\int_0^x g'(v)dv = \int_0^x f_X(v)dv \text{ if } 0 < g(0 \leq v \leq x) < 1.$$

Since $0 < F_X(0 \leq v \leq x) < 1$, we see that

$$g(x) = F_X(x)$$

where F_X is the cumulative distribution function.

Example

Let X be a continuous random variable with p.d.f.:

$$f_X(x) = \frac{1}{\tau} e^{-x/\tau}$$

and c.d.f.:

$$\begin{aligned} F_X(x') &= \int_0^{x'} \frac{1}{\tau} e^{-\frac{x}{\tau}} dx \\ &= -e^{-\frac{x}{\tau}} \Big|_0^{x'} \\ &= 1 - e^{-\frac{x'}{\tau}} \end{aligned}$$

If U is a continuous random variable such that:

$$u = F_X(x) = 1 - e^{-\frac{x}{\tau}}$$

then $f_U(u)$ is uniform:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The Other Direction

Does there exist a g, which will transform a uniform random variable, U, into a random variable, X, with p.d.f., f_X?

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$u = F_X(x)$$

it follows that:

$$x = F_X^{-1}(u)$$

where F_X is the cumulative distribution function and F_X^{-1} is the *inverse cumulative distribution function.*

Example

Let U be a uniform random variable with p.d.f.:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

and let X be an exponential random variable with p.d.f.:

$$f_X(x) = \frac{1}{\tau} e^{-\frac{x}{\tau}}$$

and c.d.f.:

$$F_X(x') = 1 - e^{-\frac{x'}{\tau}},$$

then samples of u can be transformed into samples of x as follows:

$$x = F_X^{-1}(u) = -\tau \ln(1 - u).$$

Histogram Matching

Let X and Y be random variables with p.d.f.'s f_X and f_Y. Is there a function, g, which will transform samples of X so that they have the same distribution as Y? The c.d.f., F_X, transforms the random variable, X, into the uniform random variable, U:

$$u = F_X(x)$$

where

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The inverse c.d.f., F_Y^{-1}, transforms the uniform random variable, U, into the random variable, Y:

$$y = F_Y^{-1}(u).$$

Histogram Matching (contd.)

It follows that

$$y = g(x) = F_Y^{-1}(F_X(x)).$$