Histogram Equalization

Last time we derived an expression for $f_Y$ in terms of $g'$, $g^{-1}$ and $f_X$ when $Y = g(X)$:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}.$$  

A uniform random variable, $U$, has the following p.d.f.:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Does there exist a $g$, which will transform a random variable, $X$, with p.d.f., $f_X$, into a uniform random variable, $U$, with p.d.f., $f_U$?
Histogram Equalization (contd.)

We start by setting the expression for $f_Y$ equal to the expression for $f_U$:

\[
\frac{f_X(g^{-1}(u))}{g'(g^{-1}(u))} = \begin{cases} 
1 \text{ if } 0 < u < 1 \\
0 \text{ otherwise }
\end{cases}
\]

and multiplying both sides by $g'(g^{-1}(u))$

\[
f_X(g^{-1}(u)) = g'(g^{-1}(u)) \text{ if } 0 < u < 1.
\]
Histogram Equalization (contd.)

After substituting $v$ for $g^{-1}(u)$ and $g(v)$ for $u$ we have

$$g'(v) = f_X(v) \text{ if } 0 < g(v) < 1.$$  

Integrating both sides of the equation:

$$\int_0^x g'(v) dv = \int_0^x f_X(v) dv \text{ if } 0 < g(0 \leq v \leq x) < 1.$$  

Since $F'_X(v) = f_X(v)$ and $0 < F_X(0 \leq v \leq x) < 1$, we see that

$$g(x) = F_X(x)$$

where $F_X$ is the cumulative distribution function.
Example

Let $X$ be a continuous random variable with p.d.f.:

$$f_X(x) = \frac{1}{\tau}e^{-x/\tau}$$

and c.d.f.:

$$F_X(x') = \int_0^{x'} \frac{1}{\tau}e^{-x/\tau}dx$$

$$= -e^{-x'/\tau} \bigg|_0^{x'}$$

$$= 1 - e^{-x'/\tau}$$

If $U$ is a continuous random variable such that:

$$u = F_X(x) = 1 - e^{-x/\tau}$$

then $f_U(u)$ is uniform:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$
The Other Direction

Does there exist a $g$, which will transform a uniform random variable, $U$, into a random variable, $X$, with p.d.f., $f_X$?

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$u = F_X(x)$$

it follows that:

$$x = F_X^{-1}(u)$$

where $F_X$ is the cumulative distribution function and $F_X^{-1}$ is the inverse cumulative distribution function.
Example

Let $U$ be a uniform random variable with p.d.f.:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $X$ be an exponential random variable with p.d.f.:

$$f_X(x) = \frac{1}{\tau} e^{-\frac{x}{\tau}}$$

and c.d.f.:

$$F_X(x') = 1 - e^{-\frac{x'}{\tau}},$$

then samples of $u$ can be transformed into samples of $x$ as follows:

$$x = F_X^{-1}(u) = -\tau \ln(1 - u).$$
Histogram Matching

Let $X$ and $Y$ be r.v.’s with p.d.f.’s $f_X$ and $f_Y$. Is there a function, $g$, which will transform samples of $X$ so that they have the same distribution as $Y$?
Histogram Matching (contd.)

Recall that the c.d.f., $F_X$, transforms the r.v., $X$, into the uniform r.v., $U$:

$$u = F_X(x)$$

where

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise}. \end{cases}$$

The inverse c.d.f., $F_Y^{-1}$, transforms the uniform r.v., $U$, into the r.v., $Y$:

$$y = F_Y^{-1}(u).$$

It follows that

$$y = g(x) = F_Y^{-1}(F_X(x)).$$
Figure 1: The slope of the c.d.f. $F_X$ is equal to the p.d.f. $f_X$. Where the slope exceeds one, the c.d.f. dilates the p.d.f., decreasing its density. Conversely, where the slope is less than one, the c.d.f. contracts the p.d.f., increasing its density.
Figure 2: Samples of $X$ are transformed to samples of a uniform random variable $U$ using $F_X$. Samples of the $U$ are then transformed to samples of $Y$ using $F_Y^{-1}$. 