Uniform Random Variable

A *uniform random variable*, U, has the following p.d.f.:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Rejection Method

Given a source of uniform random values, a sample of a continuous r.v. X with p.d.f. f_X can be generated as follows:

1. Randomly choose a value x from the domain of a continuous r.v. X.
2. Generate a sample u of the uniform r.v. U.
3. If $u \leq f_X(x)$ then accept x, otherwise reject x and go back to Step 1.

Histogram Equalization

Last week we derived an expression for f_Y in terms of g', g^{-1} and f_X when $Y = g(X)$:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}.$$

Does there exist a g, which will transform a random variable, X, with p.d.f., f_X, into a uniform random variable, U, with p.d.f., f_U?

Histogram Equalization (contd.)

We start by setting the expression for f_Y equal to the expression for f_U:

$$\frac{f_X(g^{-1}(u))}{g'(g^{-1}(u))} = \begin{cases} 1 \text{ if } 0 < u < 1 \\ 0 \text{ otherwise.} \end{cases}$$

Multiplying both sides by $g'(g^{-1}(u))$ yields

$$f_X(g^{-1}(u)) = \begin{cases} g'(g^{-1}(u)) & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Histogram Equalization (contd.)

After substituting v for $g^{-1}(u)$ and $g(v)$ for u we have

$$g'(v) = \begin{cases} f_X(v) & \text{if } 0 < g(v) < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Integrating both sides yields

$$g(x) = \begin{cases} \int_0^x f_X(v)dv & \text{if } 0 < g(0 \leq v \leq x) < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\int_0^x f_X(v)dv = F_X(x)$ and $0 < F_X(0 \leq v \leq x) < 1$, we see that

$$g(x) = F_X(x)$$

where F_X is the cumulative distribution function.

Example

Let X be a continuous random variable with p.d.f.:

$$f_X(x) = \frac{1}{\tau} e^{-x/\tau}$$

and c.d.f.:

$$\begin{aligned} F_X(x') &= \int_0^{x'} \frac{1}{\tau} e^{-\frac{x}{\tau}} dx \\ &= -e^{-\frac{x}{\tau}} \Big|_0^{x'} \\ &= 1 - e^{-\frac{x'}{\tau}} \end{aligned}$$

If U is a continuous random variable such that:

$$u = F_X(x) = 1 - e^{-\frac{x}{\tau}}$$

then $f_U(u)$ is uniform:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Transformation Method

Does there exist a g, which will transform a uniform random variable, U, into a random variable, X, with p.d.f., f_X?

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$u = F_X(x)$$

it follows that:

$$x = F_X^{-1}(u)$$

where F_X is the cumulative distribution function and F_X^{-1} is the *inverse cumulative distribution function.*

Example

Let U be a uniform random variable with p.d.f.:

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise} \end{cases}$$

and let X be an exponential random variable with p.d.f.:

$$f_X(x) = \frac{1}{\tau} e^{-\frac{x}{\tau}}$$

and c.d.f.:

$$u = F_X(x) = 1 - e^{-\frac{x}{\tau}},$$

then samples of u can be transformed into samples of x as follows:

$$x = F_X^{-1}(u) = -\tau \ln(1-u).$$

Histogram Matching

Let X and Y be r.v.'s with p.d.f.'s f_X and f_Y. Is there a function, g, which will transform samples of X so that they have the same distribution as Y?

Histogram Matching (contd.)

Recall that the c.d.f., F_X, transforms the r.v., X, into the uniform r.v., U:

$$u = F_X(x)$$

where

$$f_U(u) = \begin{cases} 1 & \text{if } 0 < u < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The inverse c.d.f., F_Y^{-1}, transforms the uniform r.v., U, into the r.v., Y:

$$y = F_Y^{-1}(u).$$

It follows that

$$y = g(x) = F_Y^{-1}(F_X(x)).$$

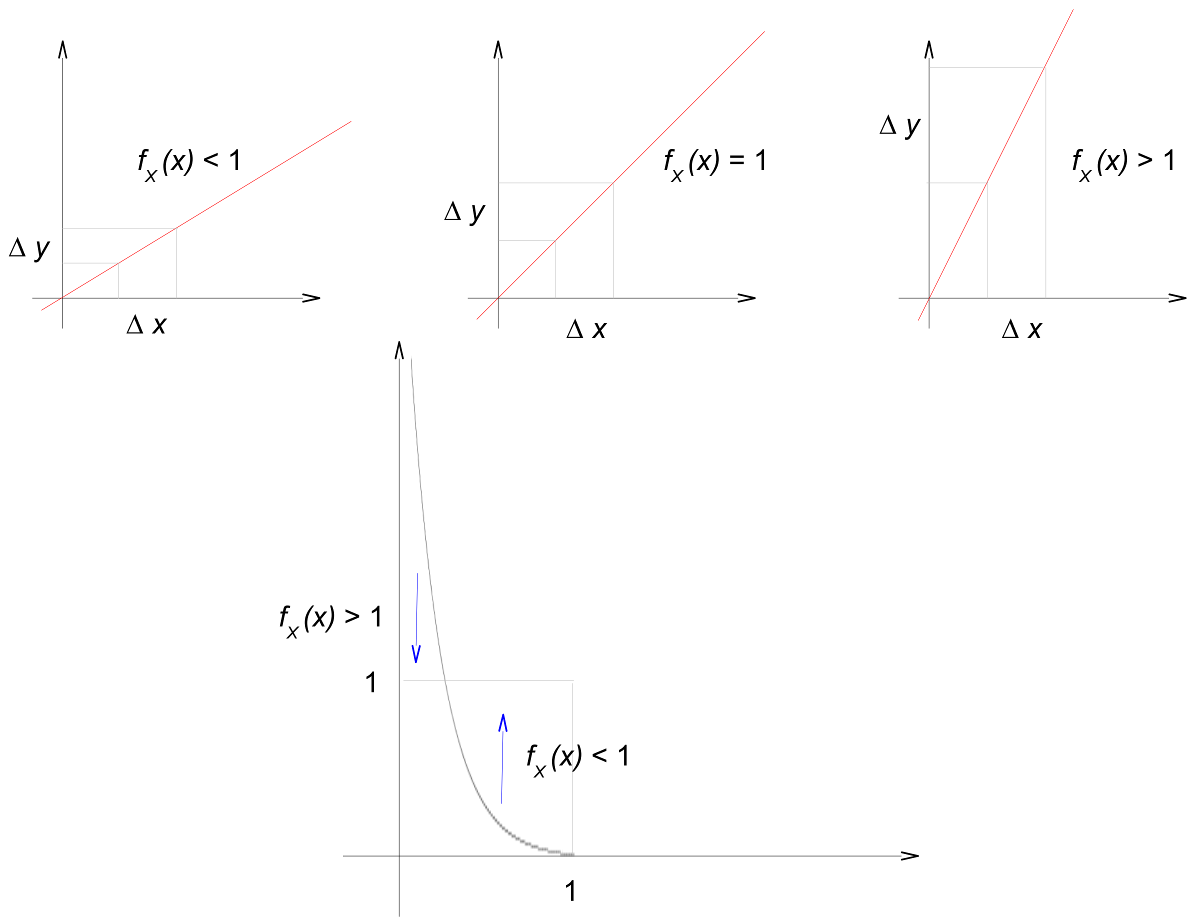

Figure 1: The slope of the c.d.f. F_X is equal to the p.d.f. f_X. Where the slope exceeds one, the c.d.f. dilates the p.d.f., decreasing its density. Conversely, where the slope is less than one, the c.d.f. contracts the p.d.f., increasing its density.

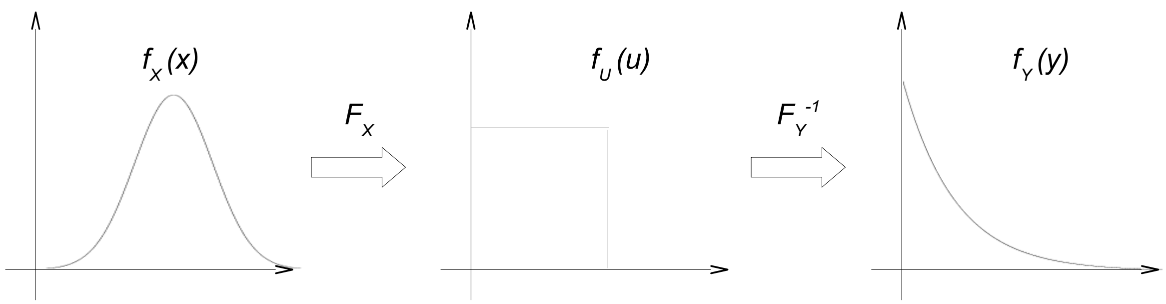

Figure 2: Samples of X are transformed to samples of a uniform random variable U using F_X. Samples of U are then transformed to samples of Y using F_Y^{-1}.