Frames vs. Bases

- A set of vectors form a *basis* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$ and are linearly independent.

- A set of $N \geq M$ vectors form a *frame* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$. 
Basis Matrix

Let $\mathcal{B}$ consist of the $M$ basis vectors, $\mathbf{b}_1 \ldots \mathbf{b}_N \in \mathbb{R}^M$. Let $\mathbf{x} \in \mathbb{R}^M$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\mathcal{B}$. It follows that

$$\mathbf{y} = x_1 \mathbf{b}_1 + x_2 \mathbf{b}_2 + \cdots + x_M \mathbf{b}_M.$$ 

This is just the matrix vector product

$$\mathbf{y} = \mathbf{B} \mathbf{x}$$

where the basis matrix, $\mathbf{B}$, is the $M \times M$ matrix,

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_M \end{bmatrix}.$$
Inverse Basis Matrix

To find the representation of the vector $y$ in the basis $\mathcal{B}$ we multiply $y$ by $B^{-1}$:

$$x = B^{-1}y.$$  

The components of the representation of $y$ in $\mathcal{B}$ are inner products of $y$ with the rows of $B^{-1}$. The transposes of these row vectors form a dual basis $\tilde{\mathcal{B}}$. 
Figure 1: Primal $\mathcal{B}$ (right) and dual $\tilde{\mathcal{B}}$ (left) bases and standard basis (center). The vectors which comprise $\tilde{\mathcal{B}}$ are the transposes of the rows of $\mathbf{B}^{-1}$. 
Frame Matrix

Let $\mathcal{F}$ consist of the $N$ frame vectors, $f_1 \ldots f_N \in \mathbb{R}^M$, where $N \geq M$. Let $x \in \mathbb{R}^N$ be a representation of $y \in \mathbb{R}^M$ in $\mathcal{F}$. It follows that

$$\mathbf{y} = x_1 f_1 + x_2 f_2 + \cdots + x_N f_N.$$  

This is just the matrix vector product

$$\mathbf{y} = \mathbf{F} \mathbf{x}$$

where the *frame matrix*, $\mathbf{F}$, is the $M \times N$ matrix,

$$\mathbf{F} = \begin{bmatrix} f_1 & f_2 & \ldots & f_N \end{bmatrix}.$$
Inverse Frame Matrix (contd.)

We might guess that

\[ x = F^{-1}y \]

where \( FF^{-1} = I \). Unfortunately, because \( F \) is not square, it has no simple inverse. However, it has an infinite number of *right-inverses*. Each of the \( x \) produced when \( y \) is multiplied by a distinct right-inverse is a distinct representation of the vector \( y \) in the frame, \( F \).
Pseudoinverse

We observe that the pseudoinverse

\[ F^+ = F^T (FF^T)^{-1} \]

is a right-inverse of \( F \). We call the \( N \times M \) matrix, \( F^+ \), an inverse frame matrix because it maps vectors, \( y \in \mathbb{R}^M \), into representations, \( x \in \mathbb{R}^N \).
Frame Bounds

Let $\mathcal{F}$ consist of the $N$ frame vectors, $f_1 \ldots f_N \in \mathbb{R}^M$, where $N \geq M$, and let $F^+$ be the inverse frame matrix. $\mathcal{F}$ is a frame iff for all $y \in \mathbb{R}^M$ there exist $A$ and $B$ where $0 < A \leq B < \infty$ and where

$$\frac{1}{B} \|y\|^2 \leq \|F^+y\|^2 \leq \frac{1}{A} \|y\|^2.$$ 

$A$ and $B$ are called the frame bounds.
Dual Frame

If $\mathcal{F}$ consists of the $N$ frame vectors, $\mathbf{f}_1\ldots\mathbf{f}_N \in \mathbb{R}^M$, with inverse frame matrix $\mathbf{F}^+$, then the dual frame, $\mathcal{F}$, consists of the $N$ frame vectors, $\tilde{\mathbf{f}}_1\ldots\tilde{\mathbf{f}}_N \in \mathbb{R}^M$:

$$(\mathbf{F}^+)^\top = \begin{bmatrix} \tilde{\mathbf{f}}_1 & \tilde{\mathbf{f}}_2 & \ldots & \tilde{\mathbf{f}}_N \end{bmatrix}.$$ 

Let $\tilde{\mathbf{x}} \in \mathbb{R}^N$ be a representation of $\mathbf{y} \in \mathbb{R}^M$ in $\mathcal{F}$. It follows that

$$\mathbf{y} = (\mathbf{F}^+)^\top \tilde{\mathbf{x}}.$$ 

Consequently, $(\mathbf{F}^+)^\top$ is the frame matrix for the dual frame, $\mathcal{F}$. 
The vectors which comprise $\tilde{F}$ are the transposes of the rows of $F^+$. 

Figure 2: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) frames and standard basis (center).
Dual Frame (contd.)

Because $F^T$ is a right inverse of $(F^+)^T$:

$$(F^+)^T F^T = I.$$ 

It follows that $F^T$ is the inverse frame matrix for the dual frame, $\tilde{F}$, and

$$A\|y\|^2 \leq \|F^Ty\|^2 \leq B\|y\|^2,$$

for all $y \in \mathbb{R}^M$. 
Example

What is the representation of $y = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ in the frame formed by the vectors $f_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$, $f_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$ and $f_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$?

$F = \begin{bmatrix} 0.70711 & -0.70711 & 0 \\ 0.70711 & 0.70711 & -1 \end{bmatrix}$

$F^+ = \begin{bmatrix} 0.70711 & 0.35355 \\ -0.70711 & 0.35355 \\ 0 & -0.5 \end{bmatrix}$

$F^+ y = \begin{bmatrix} 1.06066 \\ -0.35355 \\ -0.5 \end{bmatrix}$
Tight-Frames

If $A = B$ then

$$||F^T y||^2 = A||y||^2$$

and $F$ is said to be a tight-frame. When $F$ is a tight-frame,

$$F^+ = \frac{1}{A} F^T.$$

If $||f_i|| = 1$ for all frame vectors, $f_i$, then $A$ equals the overcompleteness of the representation. When $A = B = 1$, then $F$ is an orthonormal basis and $F = \tilde{F}$. 
Figure 3: Primal $\mathcal{F}$ (right) and dual $\mathcal{F}'$ (left) tight-frames with overcompleteness two and standard basis (center).
Example

What is the representation of \( y = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \) in the frame formed by the vectors \( f_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, f_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, f_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T \) and \( f_4 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T \)?

\[
\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}
\]

\[
\mathbf{F}^+ = \frac{1}{2} \mathbf{F}^T = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \\ 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}
\]

\[
\frac{1}{2} \mathbf{F}^T y = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}
\]
Figure 4: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness one (orthonormal bases) and standard basis (center).
Summary of Notation

- $y \in \mathbb{R}^M$ – a vector.
- $x \in \mathbb{R}^N$ – a representation of $y$ in $\mathcal{F}$.
- $f_1 \ldots f_N \in \mathbb{R}^M$ where $N \geq M$ – frame vectors for $\mathcal{F}$.
- $F = \begin{bmatrix} f_1 & f_2 & \ldots & f_N \end{bmatrix}$ – frame matrix for $\mathcal{F}$.
- $F : \mathbb{R}^N \rightarrow \mathbb{R}^M$.
- $F^+ = F^T (F^T F)^{-1}$ – inverse frame matrix for $\mathcal{F}$.
- $F^+ : \mathbb{R}^M \rightarrow \mathbb{R}^N$.
- $0 < A \leq B < \infty$ – bounds for $\mathcal{F}$. 
Summary of Notation (contd.)

• \( \tilde{x} \in \mathbb{R}^M \) – a representation of \( y \) in \( \tilde{F} \).
• \( \tilde{f}_1 \ldots \tilde{f}_N \in \mathbb{R}^M \) – frame vectors for \( \tilde{F} \).
• \( (F^+)^T = \begin{bmatrix} \tilde{f}_1 & \tilde{f}_2 & \ldots & \tilde{f}_N \end{bmatrix} \) – frame matrix for \( \tilde{F} \).
• \( (F^+)^T : \mathbb{R}^N \rightarrow \mathbb{R}^M \).
• \( F^T \) – inverse frame matrix for \( \tilde{F} \).
• \( F^T : \mathbb{R}^M \rightarrow \mathbb{R}^N \).
• \( 0 < \frac{1}{B} \leq \frac{1}{A} < \infty \) – bounds for \( \tilde{F} \).