DCT Basis Functions

Figure 1: Basis functions of Discrete Cosine Transform (DCT).
Simple Cell Receptive Fields

Figure 2: Cosine (left) and sine gratings (right) in Gaussian envelopes, known as \textit{Gabor functions}, closely resemble the receptive fields of simple cells in primary visual cortex (V1). Gabor functions at a range of scales and orientations are centered at all positions $(x,y)$ in the visual field.

Figure 3: Cosine Gabor functions of different scales, log $r$, and orientations, $\theta$. 
Frames vs. Bases

• A set of vectors form a *basis* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$ and are linearly independent.

• A set of $N \geq M$ vectors form a *frame* for $\mathbb{R}^M$ if they span $\mathbb{R}^M$. 
Advantages of Frame Representations

- Using bases $\mathcal{B}$ it is possible to build sparse, invertible representations.
- Using frames $\mathcal{F}$ it is possible to build sparse, invertible representations that are also *Euclidean equivariant.*
Euclidean Equivariance

Primary visual cortex uses a frame operator $\mathcal{F}$ to transform an input representation, $I : \mathbb{R}^2 \to \mathbb{R}$, into an output representation of higher dimensionality, $O : \mathbb{R}^2 \times \mathbb{R}^+ \times S^1 \to \mathbb{R}$:

$$I \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \xrightarrow{\mathcal{F}} O \left( \begin{bmatrix} x \\ y \\ \log r \\ \theta \end{bmatrix} \right).$$

A Euclidean transformation, $\mathcal{T}$, takes an input representation and returns the same representation rotated, translated and scaled:

$$I \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \xrightarrow{\mathcal{T}} I \left( \begin{bmatrix} s(x \cos \phi + y \sin \phi) + u \\ s(y \sin \phi - x \cos \phi) + v \end{bmatrix} \right).$$
Euclidean Equivariance (contd.)

An operator, $\mathcal{F}$, is Euclidean equivariant, iff it commutes with $\mathcal{T}$. This property can be depicted using a commutative diagram:

\[
\begin{array}{ccc}
I & \xrightarrow{\mathcal{T}} & \mathcal{T}I \\
\downarrow & & \downarrow \mathcal{F} \\
\mathcal{F}I & \xrightarrow{\mathcal{T}'} & O
\end{array}
\]

where $\mathcal{T}'$ is the corresponding transformation of the output representation of higher dimensionality:

\[
O \left( \begin{bmatrix} x \\ y \\ \log r \\ \theta \end{bmatrix} \right) \xrightarrow{\mathcal{T}'} O \left( \begin{bmatrix} s(x \cos \phi + y \sin \phi) + u \\ s(y \sin \phi - x \cos \phi) + v \\ \log r + \log s \\ \theta + \phi \end{bmatrix} \right).
\]
Synthesis Matrix

Let $\mathcal{B}$ consist of the $M$ basis vectors, $b_1 \ldots b_M \in \mathbb{R}^M$. Let $\{y\}_\mathcal{B} \in \mathbb{R}^M$ be a representation of $y \in \mathbb{R}^M$ in $\mathcal{B}$. It follows that

$$y = \mathbf{B}\{y\}_\mathcal{B}$$

where the *synthesis matrix*, $\mathbf{B}$, is the $M \times M$ matrix,

$$\mathbf{B} = \begin{bmatrix} b_1 & b_2 & \ldots & b_M \end{bmatrix}.$$ 

where $b_i$ is column $i$ of $\mathbf{B}$. 
Analysis Matrix

To find the representation of the vector $y$ in the basis $\mathcal{B}$ we multiply $y$ by the analysis matrix $\mathbf{B}^{-1}$:

$$\{y\}_\mathcal{B} = \mathbf{B}^{-1} y.$$

The components of the representation of $y$ in $\mathcal{B}$ are inner products of $y$ with the rows of $\mathbf{B}^{-1}$:

$$\mathbf{B}^{-1} = \begin{bmatrix}
\tilde{b}_1^T \\
\tilde{b}_2^T \\
\vdots \\
\tilde{b}_M^T
\end{bmatrix}.$$

where $\tilde{b}_i^T$ is row $i$ of $\mathbf{B}^{-1}$. 
Dual Basis

The transposes of these row vectors form a dual basis, $\tilde{B}$, with synthesis matrix:

$$(B^{-1})^T = [\tilde{b}_1 | \tilde{b}_2 | \ldots | \tilde{b}_M]$$

and analysis matrix:

$$B^T = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_M^T \end{bmatrix}.$$ 

The relationship between the vectors of the primal ($B$) and dual ($\tilde{B}$) bases is:

$$\langle b_i, \tilde{b}_j \rangle = \delta_{ij}.$$ 

The biorthogonality of the columns of $B$ and the rows of $B^{-1}$ follows immediately from the definition of matrix inverse.
Recall that any $N \times N$ matrix, $P$, with $N$ distinct eigenvalues, $\lambda_i$, can be factored into a product of three matrices:

$$P = X \Lambda Y^T$$

where the columns of

$$X = \begin{bmatrix} x_1 & x_2 & \ldots & x_M \end{bmatrix}$$

are right eigenvectors satisfying $\lambda_i x_i = P x_i$ and the rows of

$$Y^T = \begin{bmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_M^T \end{bmatrix}$$

are left eigenvectors satisfying $\lambda_i y_i^T = y_i^T P$ and $\Lambda$ is a diagonal matrix of eigenvalues where $\Lambda_{ii} = \lambda_i$. 
Example (contd.)

Because $X$ and $Y^T$ are inverses:

$$\langle x_i, y_j \rangle = \delta_{ij}.$$ 

Consequently, the right and left eigenvectors form primal basis $X$ and dual basis $Y$. We take inner products with the left eigenvectors $Y$ to find the representation in the basis of right eigenvectors $X$ and vice versa.
Figure 4: Primal $\mathcal{B}$ (right) and dual $\tilde{\mathcal{B}}$ (left) bases and standard basis (center). The vectors which comprise $\tilde{\mathcal{B}}$ are the transposes of the rows of $\mathbf{B}^{-1}$. 
Frame Synthesis Matrix

Let $\mathcal{F}$ consist of the $N$ frame vectors, $f_1 \ldots f_N \in \mathbb{R}^M$, where $N \geq M$. Let $\{y\}_\mathcal{F} \in \mathbb{R}^N$ be a representation of $y \in \mathbb{R}^M$ in $\mathcal{F}$. It follows that

$$y = F \{y\}_\mathcal{F}$$

where the synthesis matrix, $F$, is the $M \times N$ matrix,

$$F = \begin{bmatrix} f_1 & f_2 & \ldots & f_N \end{bmatrix}.$$
Frame Analysis Matrix

We might guess that

\[ \{y\}_\mathcal{F} = F^{-1}y \]

where \( F^{-1} \) is \( N \times M \) and \( FF^{-1} = I \). Unfortunately, because \( F \) is not square, there is no unique inverse. However, \( F \) has an infinite number of \textit{right-inverses}. Each of the \( \{y\}_\mathcal{F} \) produced when \( y \) is multiplied by a distinct right-inverse is a distinct representation of the vector \( y \) in the frame, \( \mathcal{F} \).
Pseudoinverse

The *pseudoinverse* of $F$ is

$$F^+ = F^T (FF^T)^{-1}.$$ 

$F^+$ is a right inverse of $F$ because

$$FF^+ = FF^T (FF^T)^{-1} = I.$$ 

The $N \times M$ matrix, $F^+$, is as an *analysis matrix* because it transforms a representation $y \in \mathbb{R}^M$, in the standard basis, into a representation $\{y\}_F \in \mathbb{R}^N$, in the frame, $F$:

$$\{y\}_F = F^+ y.$$
Dual Frame and Its Synthesis Matrix

If $\mathcal{F}$ consists of the $N$ frame vectors, $f_1 \ldots f_N \in \mathbb{R}^M$, with analysis matrix $F^+$, then there exists a dual frame, $\tilde{\mathcal{F}}$, consisting of the $N$ frame vectors, $\tilde{f}_1 \ldots \tilde{f}_N \in \mathbb{R}^M$:

$$(F^+)^T = [\tilde{f}_1 | \tilde{f}_2 | \ldots | \tilde{f}_N].$$

Let $\{y\}_{\tilde{\mathcal{F}}} \in \mathbb{R}^N$ be a representation of $y \in \mathbb{R}^M$ in $\tilde{\mathcal{F}}$. It follows that

$$y = (F^+)^T \{y\}_{\tilde{\mathcal{F}}},$$

and $(F^+)^T$ is the synthesis matrix for the dual frame, $\tilde{\mathcal{F}}$. 

Because $\mathbf{F}\mathbf{F}^+ = \mathbf{I}$, it follows that $\mathbf{F}^T$ is a right inverse of $(\mathbf{F}^+)^T$:

$$(\mathbf{F}^+)^T \mathbf{F}^T = \mathbf{I}.$$  

Consequently, $\mathbf{F}^T$ is an analysis matrix for the dual frame, $\widetilde{\mathcal{F}}$:

$$\{\mathbf{y}\}_{\widetilde{\mathcal{F}}} = \mathbf{F}^T \mathbf{y}.$$
Span of Dual Frame

The $N$ vectors $\tilde{\mathcal{F}}$ form a frame for $\mathbb{R}^M$ iff for every $y \in \mathbb{R}^M$ of finite non-zero length there is a finite non-zero length representation of $y$ in $\tilde{\mathcal{F}}$:

$$A||y||^2 \leq ||\{y\}_{\tilde{\mathcal{F}}}||^2 \leq B||y||^2$$

where $0 < A \leq B < \infty$. 
Span of Primal Frame

Because the spans of the primal ($\mathcal{F}$) and dual ($\tilde{\mathcal{F}}$) frames are the same, and because

$$\{y\}_{\tilde{\mathcal{F}}} = \mathbf{F}^T y$$

$\mathcal{F}$ is a frame iff for all $y \in \mathbb{R}^M$ there exist $A$ and $B$ where $0 < A \leq B < \infty$ and where

$$A\|y\|^2 \leq \|\mathbf{F}^T y\|^2 \leq B\|y\|^2.$$  

$A$ and $B$ are called the frame bounds. This is significant because this is a necessary and sufficient condition for a set of vectors (the columns of $\mathbf{F}$) to form a frame.
Figure 5: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) frames and standard basis (center). The vectors which comprise $\tilde{\mathcal{F}}$ are the transposes of the rows of $F^+$. 
Example

What is the representation of \( y = [1 \ 1]^T \) in the frame formed by the vectors \( f_1 = \left[ \frac{\sqrt{2}}{2} \ \frac{\sqrt{2}}{2} \right]^T \), \( f_2 = \left[ -\frac{\sqrt{2}}{2} \ \frac{\sqrt{2}}{2} \right]^T \) and \( f_3 = \left[ 0 \ -1 \right]^T \)?

\[
F = \begin{bmatrix}
0.70711 & -0.70711 & 0 \\
0.70711 & 0.70711 & -1
\end{bmatrix}
\]

\[
F^+ = \begin{bmatrix}
0.70711 & 0.35355 \\
-0.70711 & 0.35355 \\
0 & -0.5
\end{bmatrix}
\]

\[
F^+ y = \begin{bmatrix}
1.06066 \\
-0.35355 \\
-0.5
\end{bmatrix}
\]
Tight-Frames

If $A = B$ then

$$\|F^Ty\|^2 = A\|y\|^2$$

and $\mathcal{F}$ is said to be a tight-frame. When $\mathcal{F}$ is a tight-frame,

$$F^+ = \frac{1}{A}F^T.$$ 

If $\|f_i\| = 1$ for all frame vectors, $f_i$, then $A$ equals the overcompleteness of the representation. When $A = B = 1$, then $\mathcal{F}$ is an orthonormal basis and $\mathcal{F} = \tilde{\mathcal{F}}$. 
Figure 6: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness two and standard basis (center).
Example

What is the representation of $y = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ in the frame formed by the vectors $f_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $f_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $f_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$ and $f_4 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$?

$$F = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$F^+ = \frac{1}{2} F^T = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \\ 0 & -0.5 \\ -0.5 & 0 \end{bmatrix}$$

$$\frac{1}{2} F^T y = \begin{bmatrix} 0.5 \\ 0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$
Figure 7: Primal $\mathcal{F}$ (right) and dual $\tilde{\mathcal{F}}$ (left) tight-frames with overcompleteness one (orthonormal bases) and standard basis (center).
Summary of Notation

• $y \in \mathbb{R}^M$ – a vector.

• $\{y\}_\mathcal{F} \in \mathbb{R}^N$ – a representation of $y$ in primal frame $\mathcal{F}$.

• $f_1 \ldots f_N \in \mathbb{R}^M$ where $N \geq M$ – frame vectors for primal frame $\mathcal{F}$.

• $F = \begin{bmatrix} f_1 & f_2 & \ldots & f_N \end{bmatrix}$ – synthesis matrix for primal frame $\mathcal{F}$.

• $F : \mathbb{R}^N \to \mathbb{R}^M$.

• $F^+ = F^T (F^T F)^{-1}$ – analysis matrix for primal frame $\mathcal{F}$.

• $F^+ : \mathbb{R}^M \to \mathbb{R}^N$.

• $0 < A \leq B < \infty$ – bounds for primal frame $\mathcal{F}$. 
Summary of Notation (contd.)

- $\{y\}_{\tilde{F}} \in \mathbb{R}^M$ – a representation of $y$ in dual frame $\tilde{F}$.
- $\tilde{f}_1 \ldots \tilde{f}_N \in \mathbb{R}^M$ – frame vectors for dual frame $\tilde{F}$.
- $(F^+)^T = \begin{bmatrix} \tilde{f}_1 & \tilde{f}_2 & \ldots & \tilde{f}_N \end{bmatrix}$ – synthesis matrix for dual frame $\tilde{F}$.
- $(F^+)^T : \mathbb{R}^N \to \mathbb{R}^M$.
- $F^T$ – analysis matrix for dual frame $\tilde{F}$.
- $F^T : \mathbb{R}^M \to \mathbb{R}^N$.
- $0 < \frac{1}{B} \leq \frac{1}{A} < \infty$ – bounds for dual frame $\tilde{F}$. 