Functions of Random Variables

Let's consider a continuous r.v., Y, which is a differentiable, increasing function of a second continuous r.v., X:

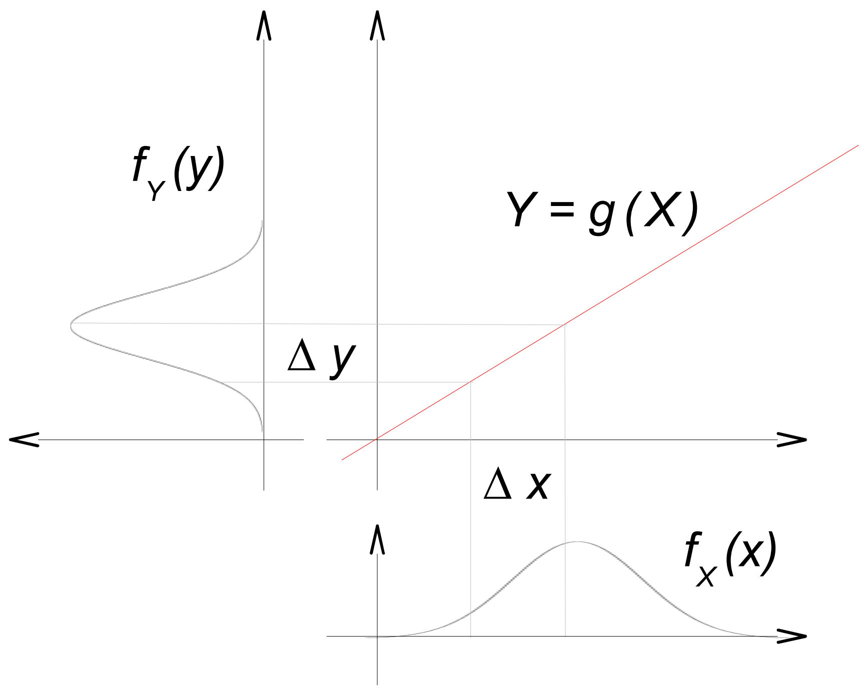
$$Y = g(X).$$

Because g is differentiable and increasing, g' and g^{-1} are guaranteed to exist. Because g maps all $x \leq s \leq x + \Delta x$ to $y \leq t \leq y + \Delta y$:

$$\int_x^{x+\Delta x} f_X(s)ds = \int_y^{y+\Delta y} f_Y(t)dt.$$

It follows that for small Δx:

$$f_Y(y)\Delta y \approx f_X(x)\Delta x.$$



Dividing by Δy, we get an approximate expression for f_Y in terms of f_X:

$$f_Y(y) \approx f_X(x)\frac{\Delta x}{\Delta y}.$$

Functions of Random Variables (contd.)

This is exact in the limit $\Delta x \to 0$:

$$\begin{aligned} f_Y(y) &= \lim_{\Delta x \to 0} f_X(x) \frac{\Delta x}{\Delta y} \\ &= \lim_{\Delta x \to 0} f_X(x) \frac{1}{\Delta y / \Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f_X(x)}{\Delta y / \Delta x}. \end{aligned}$$

From calculus we know that:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = g'(x).$$

Consequently

$$f_Y(y) = \frac{f_X(x)}{g'(x)}.$$

Functions of Random Variables (contd.)

Substituting $g^{-1}(y)$ for x:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

yields an expression for f_Y in terms of g', g^{-1}, and f_X.

Linear Example

Consider a continuous r.v., Y, which is a linear function of a continuous r.v., X. Specifically, $Y = aX + b$. It follows that

$$\begin{aligned} g(x) &= ax + b \\ g'(x) &= a \\ g^{-1}(y) &= (y-b)/a. \end{aligned}$$

Substituting the above into

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

yields

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

Linear Example (contd.)

Let $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}$, then

$$f_Y(y) = \frac{1}{a\sqrt{2\pi}} e^{-\left(\frac{y-b}{a}-\mu\right)^2/2}.$$

Linear Example (contd.)

Unfortunately, there is a problem. When $a = -1$, the function g is not increasing (it is decreasing). Consequently,

$$-f_Y(y)\Delta y \approx f_X(x)\Delta x.$$

It follows that for decreasing functions,

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{-g'(g^{-1}(y))}.$$

However, we can derive a function which is correct in both cases by replacing $g'(.)$ with $|g'(.)|$ in the expression relating $f_Y(.)$ and $f_X(.)$:

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}.$$

Quadratic Example

Consider a continuous r.v., Y, which is a quadratic function of a continuous r.v., X. Specifically, $Y = X^2$. It follows that

$$\begin{aligned} g(x) &= x^2 \\ g'(x) &= 2x \\ g^{-1}(y) &= \sqrt{y}. \end{aligned}$$

Substituting the above into

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$$

yields

$$f_Y(y) = \frac{f_X(\sqrt{y})}{|2\sqrt{y}|}.$$

Quadratic Example (contd.)

Let $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, then

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{|2\sqrt{y}|}.$$

Unfortunately, there is a problem:

$$\int_0^\infty f_Y(y) dy = \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{|2\sqrt{y}|} dy = \frac{1}{2} \neq 1.$$

Can anyone see the mistake?

Quadratic Example (contd.)

The mistake is that two different values of Y satisfy $Y = X^2$:

$$g^{-1}(y) = \pm\sqrt{y}.$$

We decided to use the positive square root arbitrarily and ignored the negative square root. Hence the factor of two error. In general, if a function does not have a unique inverse, we must sum over all possible inverse values:

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(g_i^{-1}(y))}{|g'(g_i^{-1}(y))|}.$$

Quadratic Example (contd.)

Let $g_1^{-1}(y) = \sqrt{y}$ and $g_2^{-1}(y) = -\sqrt{y}$, then

$$f_Y(y) = \frac{f_X(g_1^{-1}(y))}{|g'(g_1^{-1}(y))|} + \frac{f_X(g_2^{-1}(y))}{|g'(g_2^{-1}(y))|}$$
$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-y/2}}{\sqrt{y}}.$$

This is an example of the χ^2 distribution.

Kinetic Energy

Recall from physics, that the *kinetic energy*, K, of a moving body is given by

$$K = \frac{1}{2} m V^2$$

where m is mass and V is velocity. Let V be a normally distributed random variable with mean, μ, and variance, σ^2:

$$f_V(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(v-\mu)^2/2\sigma^2}.$$

We would like to compute $f_K(k)$, the p.d.f for the continuous random variable, K.

Kinetic Energy (contd.)

We start by computing, g', g_1^{-1}, and g_2^{-1}:

$$\begin{aligned} g(v) &= \frac{1}{2}mv^2 \\ g'(v) &= mv \\ g_1^{-1}(k) &= \sqrt{2k/m} \\ g_2^{-1}(k) &= -\sqrt{2k/m}. \end{aligned}$$

Substituting $f_V(v)$ and the above into

$$f_K(k) = \frac{f_V(g_1^{-1}(k))}{|g'(g_1^{-1}(k))|} + \frac{f_V(g_2^{-1}(k))}{|g'(g_2^{-1}(k))|}$$

yields

$$\begin{aligned} f_K(k) &= \frac{e^{-\left(\sqrt{2k/m}-\mu\right)^2/2\sigma^2} + e^{-\left(-\sqrt{2k/m}-\mu\right)^2/2\sigma^2}}{\sigma\sqrt{2\pi}\, m\sqrt{2k/m}} \\ &= \frac{e^{-\left(\sqrt{2k/m}-\mu\right)^2/2\sigma^2} + e^{-\left(\sqrt{2k/m}+\mu\right)^2/2\sigma^2}}{2\sigma\sqrt{\pi\, k\, m}}. \end{aligned}$$