CS 530: Geometric and Probabilistic Methods in Computer Science
Homework 4 (Fall '06)

1. Characterize the following as linear, or non-linear, and as even, odd, or neither, and prove your characterization.

   (a) $f(x) = -x$

   (b) $f(x) = 7x + 1$

   (c) $f(x) = x^2 - x$

   (d) $f(x) = e^{-x^2}$

   (e) $f(x) = \cos x$.

2. Write a function in MATLAB which, given an $N \times 1$ vector, $\mathbf{x}$, will return an $N \times N$ circulant matrix:

$$\mathbf{A} = \left[\begin{array}{c|c|c|c} \mathbf{S}^0\mathbf{x} & \mathbf{S}^1\mathbf{x} & \cdots & \mathbf{S}^{N-1}\mathbf{x} \end{array} \right]$$

   where $S^n_{ij} = \delta(i - j - n \mod N)$ and δ is the Kronecker delta function.

3. Using MATLAB compute the following:

   (a) A 4×4 circulant matrix $\mathbf{A}$, where the first column of $\mathbf{A}$ is $\mathbf{x} = \left[\begin{array}{cccc} -1 & 2 & -3 & 4 \end{array} \right]^{\mathrm{T}}$.

   (b) $\mathbf{W}$, the matrix of right eigenvectors of $\mathbf{A}$, and Λ, the diagonal matrix of eigenvalues.

   (c) $\mathbf{W}^{-1}$ and $(\mathbf{W}^*)^{\mathbf{T}}$.

   (d) $\mathbf{Ay}$ where $\mathbf{y} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \right]^{\mathrm{T}}$.

   (e) $\mathbf{W}\Lambda(\mathbf{W}^*)^{\mathrm{T}}\mathbf{y}$ where $\mathbf{y} = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \right]^{\mathrm{T}}$.

   (f) A 100×100 circulant matrix $\mathbf{B}$ where the first column of $\mathbf{B}$ is $\mathbf{x} = \left[\begin{array}{ccccc} -1 & 2 & -3 & \cdots & 100 \end{array} \right]^{\mathrm{T}}$. There is no need to print it out.

   (g) Let λ be the eigenvalue of $\mathbf{B}$ with twelfth smallest magnitude. Plot the left and right eigenvectors of $\mathbf{B}$ with eigenvalue equal to λ. Note: The eigenvectors are complex. The real and imaginary parts should be plotted as functions of time.

4. Prove that $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$.

5. The impulse response function of a linear shift-invariant system is:

$$h(t) = \frac{\sin(\pi t)}{\pi t}$$

and its input is:

$$x(t) = \cos(4\pi t) + \cos(\pi t/2).$$

What is its output?

6. The impulse response function of a linear shift-invariant system is:

$$h(t) = e^{-\frac{\pi t^2}{2}}$$

and its input is:

$$x(t) = e^{j2\pi s_0 t}.$$

What is its output?

7. The sine Gabor function is the product of a sine and a Gaussian, $f(t) = e^{-\pi t^2} \sin(2\pi s_0 t)$. Give an expression for $F(s)$, the Fourier transform of $f(t)$.

8. Prove that the sum of two independent Gaussian random variables with zero mean and variances σ_1^2 and σ_2^2 is a Gaussian random variable with zero mean and variance $\sigma_1^2 + \sigma_2^2$.

9. The function, $f(t)$, is defined as:

$$f(t) = \begin{cases} 1 & \text{if } |at - b| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Give an expression for $F(s)$, the Fourier transform of $f(t)$.

10. The transfer function of a linear shift invariant system is $H(s) = 1/s$. The impulse response function, $h(t)$, is $\mathcal{F}^{-1}\{H(s)\}$. Give an expression for $g(t)$ where:

$$g(t) = \int_{-\infty}^{\infty} e^{\,j2\pi s_0 \tau} h(t - \tau)\, d\tau.$$

11. Compute the Fourier transform of $f(t) = -2\pi t\, e^{-\pi t^2} \cos(2\pi s_0 t)$. Hint: What is $\frac{d\left(e^{-\pi t^2}\right)}{dt}$?

12. Prove the following statement: If $\mathcal{F}\{f\}(s) = F(s)$ then $\mathcal{F}\{F\}(s) = f(-s)$. Hint: If $\mathcal{F}\{f\}(s) = F(s)$ then $\mathcal{F}^{-1}\{F\}(t) = f(t)$.

13. Prove that $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$

14. Write a MATLAB function which takes an integer argument, N, and computes the truncated Fourier series:

$$f_N(t) = \pi + \sum_{\substack{\omega = -N \\ \omega \neq 0}}^{N} \frac{j}{\omega} e^{j\omega t}$$

Plot the real and imaginary parts of $f_N(t)$ for $N = 1, 3, 6, 12, 24$ and 48. Plot the functions on the interval, $-4\pi \leq t \leq 4\pi$.