CS 530: Geometric and Probabilistic Methods in Computer Science
Homework 4 (Fall '06)

1. Characterize the following as linear, or non-linear, and as even, odd, or neither, and prove your characterization.

   (a) \( f(x) = -x \)
   (b) \( f(x) = 7x + 1 \)
   (c) \( f(x) = x^2 - x \)
   (d) \( f(x) = e^{-x^2} \)
   (e) \( f(x) = \cos x \).

2. Write a function in MATLAB which, given an \( N \times 1 \) vector, \( x \), will return an \( N \times N \) circulant matrix:

\[
A = \begin{bmatrix}
S^0 x & S^1 x & \cdots & S^{N-1} x
\end{bmatrix}
\]

where \( S^i_{ij} = \delta(i - j - n \mod N) \) and \( \delta \) is the Kronecker delta function.

3. Using MATLAB compute the following:

   (a) A \( 4 \times 4 \) circulant matrix \( A \), where the first column of \( A \) is \( x = \begin{bmatrix} -1 & 2 & -3 & 4 \end{bmatrix}^T \).
   (b) \( W \), the matrix of right eigenvectors of \( A \), and \( \Lambda \), the diagonal matrix of eigenvalues.
   (c) \( W^{-1} \) and \( (W^*)^T \).
   (d) \( Ay \) where \( y = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T \).
   (e) \( W \Lambda (W^*)^T y \) where \( y = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T \).
   (f) A \( 100 \times 100 \) circulant matrix \( B \) where the first column of \( B \) is \( x = \begin{bmatrix} -1 & 2 & -3 & \cdots & 100 \end{bmatrix}^T \). There is no need to print it out.
   (g) Let \( \lambda \) be the eigenvalue of \( B \) with twelfth smallest magnitude. Plot the left and right eigenvectors of \( B \) with eigenvalue equal to \( \lambda \). Note: The eigenvectors are complex. The real and imaginary parts should be plotted as functions of time.
4. Prove that \( \sin(x) = \frac{e^{ix} - e^{-ix}}{2j} \).

5. The impulse response function of a linear shift-invariant system is:

\[
h(t) = \frac{\sin(\pi t)}{\pi t}
\]

and its input is:

\[
x(t) = \cos(4\pi t) + \cos(\pi t / 2).
\]

What is its output?

6. The impulse response function of a linear shift-invariant system is:

\[
h(t) = e^{-\frac{\pi t^2}{2}}
\]

and its input is:

\[
x(t) = e^{j2\pi s_0 t}.
\]

What is its output?

7. The sine Gabor function is the product of a sine and a Gaussian, \( f(t) = e^{-\pi t^2} \sin(2\pi s_0 t) \). Give an expression for \( F(s) \), the Fourier transform of \( f(t) \).

8. Prove that the sum of two independent Gaussian random variables with zero mean and variances \( \sigma_1^2 \) and \( \sigma_2^2 \) is a Gaussian random variable with zero mean and variance \( \sigma_1^2 + \sigma_2^2 \).

9. The function, \( f(t) \), is defined as:

\[
f(t) = \begin{cases} 
1 & \text{if } |at - b| \leq \frac{1}{2} \\
0 & \text{otherwise.}
\end{cases}
\]

Give an expression for \( F(s) \), the Fourier transform of \( f(t) \).

10. The transfer function of a linear shift invariant system is \( H(s) = 1/s \). The impulse response function, \( h(t) \), is \( \mathcal{F}^{-1}\{H(s)\} \). Give an expression for \( g(t) \) where:

\[
g(t) = \int_{-\infty}^{\infty} e^{j2\pi s_0 \tau} h(t - \tau) \, d\tau.
\]

11. Compute the Fourier transform of \( f(t) = -2\pi t \, e^{-\pi t^2} \cos(2\pi s_0 t) \). Hint: What is \( \frac{d(e^{-\pi t^2})}{dt} \)?
12. Prove the following statement: If $\mathcal{F}\{f\}(s) = F(s)$ then $\mathcal{F}\{F\}(s) = f(-s)$.
   Hint: If $\mathcal{F}\{f\}(s) = F(s)$ then $\mathcal{F}^{-1}\{F\}(t) = f(t)$.

13. Prove that $\mathcal{F}^{-1}\{\mathcal{F}\{f\}\} = f$

14. Write a MATLAB function which takes an integer argument, $N$, and computes the truncated Fourier series:

   $$f_N(t) = \pi + \sum_{\omega = -N}^{N} \frac{j}{\omega} e^{j\omega t}$$

   Plot the real and imaginary parts of $f_N(t)$ for $N = 1, 3, 6, 12, 24$ and 48. Plot the functions on the interval, $-4\pi \leq t \leq 4\pi$. 