Marginal Probability Distribution

To compute $p_X(k)$, we sum $p_{XY}(i, j)$ over pairs of i and j where $i = k$:

$$
\begin{aligned}
p_X(k) &= \sum_{\{i,j \mid i=k\}} p_{XY}(k, j) \\
&= \sum_{j=-\infty}^{\infty} p_{XY}(k, j).
\end{aligned}
$$

Sum of Discrete r.v.'s

Let Z be a discrete random variable equal to the sum of the discrete random variables X and Y. To compute $p_Z(k)$, we sum $p_{XY}(i,j)$ over pairs of i and j where $i+j=k$:

$$p_Z(k) = \sum_{\{i,j \mid i+j=k\}} p_{XY}(i,j).$$

Observe that $i+j=k$ iff $j=k-i$. It follows that

$$p_Z(k) = \sum_{i=-\infty}^{\infty} p_{XY}(i,k-i)$$

If X and Y are statistically independent, then $p_{XY}(i,j) = p_X(i)p_Y(j)$. It follows that

$$p_Z(k) = \sum_{i=-\infty}^{\infty} p_X(i)p_Y(k-i).$$

Sum of Discrete r.v.'s (contd.)

The *discrete convolution* of f and g, written $f * g$, is defined to be:

$$\{f * g\}(k) = \sum_{i=-\infty}^{\infty} f(i)g(k-i).$$

Accordingly, if $Z = X + Y$, then

$$p_Z = p_X * p_Y.$$

Sum of Discrete r.v.'s (contd.)

Since $i+j=k$ iff $i=k-j$, we could just as easily have written $p_Z(k)$ as follows:

$$p_Z(k) = \sum_{j=-\infty}^{\infty} p_{XY}(k-j, j) = \sum_{j=-\infty}^{\infty} p_X(k-j) p_Y(j).$$

It follows that

$$p_X * p_Y = p_Y * p_X$$

and that convolution (like addition) is *commutative*.

Difference of Discrete r.v.'s

Let Z be a discrete random variable equal to the difference of the discrete random variables X and Y. To compute $p_Z(k)$, we sum $p_{XY}(i, j)$ over the pairs of i and j where $i - j = k$:

$$p_Z(k) = \sum_{\{i,j \mid i-j=k\}} p_{XY}(i, j).$$

Observe that $i - j = k$ iff $j = i - k$. It follows that

$$p_Z(k) = \sum_{i=-\infty}^{\infty} p_{XY}(i, i-k).$$

If X and Y are statistically independent, then $p_{XY}(i, j) = p_X(i)p_Y(j)$. It follows that

$$p_Z(k) = \sum_{i=-\infty}^{\infty} p_X(i)p_Y(i-k).$$

Difference of Discrete r.v.'s (contd).

The *discrete correlation* of f and g, is defined to be:

$$\{f * g(-(.))\}(k) = \sum_{i=-\infty}^{\infty} f(i)g(i-k).$$

Marginal Probability Density

To compute $f_X(z)$, we integrate $f_{XY}(x,y)$ along the line $x = z$:

$$f_X(z) = \int_{-\infty}^{\infty} f_{XY}(z,y)dy$$

Sum of Continuous r.v.'s

Let Z be a continuous random variable equal to the sum of the continuous random variables, X and Y. To compute $f_Z(z)$, we integrate $f_{XY}(x,y)$ along the line $x+y=z$ or $y=z-x$:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)dx.$$

If X and Y are statistically independent, then $f_{XY}(x,y) = f_X(x)f_Y(y)$. It follows that

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$

Sum of Continuous r.v.'s (contd.)

The *convolution* of f and g, written $f * g$, is defined to be

$$\{f * g\}(v) = \int_{-\infty}^{\infty} f(u)g(v-u)du.$$

Accordingly, if $Z = X + Y$, then

$$f_Z = f_X * f_Y = f_Y * f_X.$$

Difference of Continuous r.v.'s

Let Z be a continuous random variable equal to the difference of the continuous random variables, X and Y. To compute $f_Z(z)$, we integrate $f_{XY}(x,y)$ along the line where $x-y=z$ or $y=x-z$:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, x-z)dy.$$

If X and Y are statistically independent, then $f_{XY}(x,y) = f_X(x)f_Y(y)$. It follows that

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(x-z)dx.$$

Difference of Continuous r.v.'s (contd.)

The *correlation* of f and g, is defined to be

$$\{f * g(-(.))\}(v) = \int_{-\infty}^{\infty} f(u)g(u-v)du$$

Accordingly, if $Z = X - Y$, then

$$f_Z(z) = \{f_X * f_Y(-(.))\}(z)$$

Law of Large Numbers

Let X_1, X_2, ..., X_N be samples of a random variable, X. It follows that X_1, X_2, ..., X_N are independent, identically distributed (i.i.d.) random variables. Then

$$\lim_{N\to\infty} \frac{X_1 + X_2 + \cdots + X_N}{N} = \mu = \langle X \rangle \, .$$

i.e., the mean of an infinite number of samples of a random variable equals the expected value.

Central Limit Theorem

Let X_1, X_2, ..., X_N be samples of a random variable, X. It follows that X_1, X_2, ..., X_N are independent, identically distributed (i.i.d.) random variables. Then

$$\lim_{N\to\infty} P\left(\frac{X_1+X_2+\cdots+X_N-N\mu}{\sigma\sqrt{N}}\leq a\right)=$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}e^{-x^2/2}dx.$$

i.e., the sum of an infinite number of i.i.d. random variables is a random variable with Gaussian density.

Central Limit Theorem (contd.)

Multiplying numerator and denominator by $\frac{1}{N}$ yields:

$$\lim_{N\to\infty} P\left(\frac{\frac{X_1+X_2+\cdots+X_N}{N}-\mu}{\sigma/\sqrt{N}} \leq a\right) =$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx.$$