Even and Odd Functions

A function, f, is *even* (or *symmetric*) when

$$f(x) = f(-x).$$

A function, f, is *odd* (or *antisymmetric*) when

$$f(x) = -f(-x).$$

Even and Odd Functions (contd.)

Theorem 5.1 Any function can be written as a sum of even and odd functions.

Proof:

$$
\begin{aligned}
f(t) &= \frac{1}{2}\left[f(t) + \underbrace{f(-t) - f(-t)}_{0} + f(t)\right] \\
&= \underbrace{\frac{1}{2}[f(t) + f(-t)]}_{f_e} + \underbrace{\frac{1}{2}[f(t) - f(-t)]}_{f_o}
\end{aligned}
$$

f_e is even because $f_e(t) = f_e(-t)$:

$$
\begin{aligned}
f_e(t) &= f(t) + f(-t) \\
&= f(-t) + f(t) \\
&= f_e(-t).
\end{aligned}
$$

f_o is odd because $f_o(t) = -f_o(-t)$:

$$
\begin{aligned}
f_o(t) &= f(t) - f(-t) \\
&= -[f(-t) - f(t)] \\
&= -f_o(-t).
\end{aligned}
$$

Theorem 5.2 The integral of the product of an even function and an odd function is zero.
Proof:

$$\int_{-\infty}^{\infty} f_e(x) f_o(x) dx =$$

$$\int_{-\infty}^{0} f_e(x) f_o(x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx.$$

Substituting $-x$ for x in the first term and changing the order of integration yields:

$$\int_{0}^{\infty} f_e(-x) f_o(-x) dx + \int_{0}^{\infty} f_e(x) f_o(x) dx$$

$$= \int_{0}^{\infty} \left[f_e(-x) f_o(-x) + f_e(x) f_o(x) \right] dx.$$

Substituting $f_e(-x)$ for $f_e(x)$ and $-f_o(-x)$ for $f_o(x)$ yields:

$$\int_{0}^{\infty} \underbrace{\left[f_e(-x) f_o(-x) - f_e(-x) f_o(-x) \right]}_{0} dx.$$

Fourier Transform Symmetry

The Fourier transform of $f(t)$ is defined to be:

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt.$$

This can be rewritten as follows:

$$F(s) = \int_{-\infty}^{\infty} f(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} f(t) \sin(2\pi st) dt.$$

Substituting $f_e(t) + f_o(t)$ for $f(t)$ yields:

$$F(s) = \int_{-\infty}^{\infty} f_e(t) \cos(2\pi st) dt + \int_{-\infty}^{\infty} f_o(t) \cos(2\pi st) dt -$$

$$j \int_{-\infty}^{\infty} f_e(t) \sin(2\pi st) dt - j \int_{-\infty}^{\infty} f_o(t) \sin(2\pi st) dt.$$

However, the second and third terms are zero by Theorem 5.2:

$$F(s) = \int_{-\infty}^{\infty} f_e(t) \cos(2\pi st) dt - j \int_{-\infty}^{\infty} f_o(t) \sin(2\pi st) dt.$$

It follows that:

$$F(s) = F_e(s) + F_o(s).$$

Fourier Transform Symmetry (contd.)

- The Fourier transform of the even part is real:

$$\mathcal{F}\{f_e\}(s) = F_e(s) = \mathrm{Re}(F_e(s)).$$

- The Fourier transform of the even part is even:

$$\mathcal{F}\{f_e\}(s) = F_e(s) = F_e(-s).$$

- The Fourier transform of the odd part is imaginary:

$$\mathcal{F}\{f_o\}(s) = F_o(s) = \mathrm{Im}(F_o(s)).$$

- The Fourier transform of the odd part is odd:

$$\mathcal{F}\{f_o\}(s) = F_o(s) = -F_o(-s).$$

We can summarize the above four symmetries as follows:

$$\begin{aligned} F_e(s) + F_o(s) &= F_e(-s) - F_o(-s) \\ &= \left[F_e(-s) + F_o(-s)\right]^*. \end{aligned}$$

This is termed *Hermitian* (or *conjugate*) symmetry:

$$F(s) = F^*(-s).$$